Forward Bhabha Scattering – Theoretical Problems

Tord Riemann, DESY, Zeuthen

FCAL Collaboration Meeting
Tel-Aviv University, 18–19 Sep 2005

A project in collaboration with
Michal Czakon Univ. Würzburg (and Katowice)
Janusz Gluza DESY (and Katowice)

See also: • NPB(PS) 135 (2004), hep-ph/0406203
  • PRD 71 (2005), hep-ph/0412164
  • http://www-zeuthen.desy.de/theory/research/bhabha/

• What do we need? \[
\rightarrow 10^{-4} \text{ for } \frac{d\sigma}{d \cos \vartheta} \text{ at small } \vartheta
\]
• Higher Order Corrections – Status
• Summary
The Physics Needs

For more details see e.g.:
K. Mönig, "Bhabha scattering at the ILC"
talk at Mini-WS on Bhabha scattering, Univ. Karlsruhe, April 2005
/afs/ifh.de/user/m/moenig/public/www/bhabha_ilc.pdf

ILC – Need Bhabha cross-sections with 3–4 significant digits.

Why?

- **ILC**: $e^+e^- \rightarrow W^+W^-$, $f \bar{f}$ with $O(10^6)$ events $\rightarrow 10^{-3}$

- **GigaZ**: relevant physics derived from $Z \rightarrow$ hadrons, $l^+l^-$, the latter with $O(10^8)$ events $\rightarrow 10^{-4}$, the systematic errors (luminosity!) influence this

- **ILC**: $e^+e^- \rightarrow e^+e^-$, a probe for New Physics with $O(10^5)$ events/year $\rightarrow 10^{-3}$

Conclude: will need $\Delta \mathcal{L}/\mathcal{L} \approx 2 \times 10^{-4}$

The luminosity comes from very forward Bhabha scattering.
Some Kinematics

Need a cross-section prediction with 5 significant digits.

Perturbative orders:

\[
\left( \frac{\alpha}{\pi} \right) = 2 \times 10^{-3}
\]

\[
\left( \frac{\alpha}{\pi} \right)^2 = 0.6 \times 10^{-5}
\]

Kinematics:

\[
\sqrt{s} = 90 \ldots 1000 \text{ GeV}
\]

\[
\vartheta = 26 \ldots 82 \text{ mrad}
\]

\[
\cos \vartheta \sim 0.999 \, 66 \ldots 0.996 \, 64
\]

\[
T = \frac{s}{2} (1 - \beta^2 \cos \vartheta) > 1.36 \text{ GeV}_{\text{GigaZ}}, \ 42.2 \text{ GeV}_{\text{ILC500}}
\]

Conclude:

- *t*-channel exchange of $\gamma$ dominates everything else
- $m_e^2/s < m_e^2/T \leq 10^{-5} \ldots 10^{-7}$

- **Calculate:** 1-loop EWRC + 2-loop QED + corresp. bremsstrahlung
The 1-loop electroweak corrections (plus some leading higher order terms) are well-known, with rising technical precision, since about 1988/91.

Böhm, Denner, Hollik; Bardin, TR 1991 → Fig. 2004 Lorca, TR

2-loop Bhabha scattering: What to be done?

• Calculate:

\[ \sigma = \ (2 \rightarrow 2) \ + \ (2 \rightarrow 3) \ + \ (2 \rightarrow 4) \]

\[ \sigma = |\text{Born} + \text{1-loop} + \text{2-loop}|^2 \]

\[ + \ |(\text{Born} + \text{1-}\gamma) + (\text{1-loop} + \text{1-}\gamma)|^2 \]

\[ + \ |(\text{Born} + \text{2-}\gamma)|^2 \]

• Do not include: \[ |\text{2-loop}|^2 \]

\[ |(\text{1-loop} + \text{1-}\gamma)|^2 \]
Status by end of 2004

**Established:** $10^{-3}$ MC programs for LEP, ILC

Introduction to NLLBHA by Trentadue and to BHLUMI by Jadach in:
Proc. of Loops and Legs, Rheinsberg, Germany, 1996

Recent mini-review: Jadach, ”Theoretical error of luminosity cross section at LEP”, hep-ph/0306083 [1]

- **BHLUMI** v.4.04: Jadach, Placzek, Richter-Was, Was: CPC 1997
- **NLLBHA**: Arbuzov, Fadin, Kuraev, Lipatov, Merenkov, Trentadue: NPB 1997, CERN 96-01
- **SAMBHA**: Arbuzov, Haidt, Matteuzzi, Paganoni, Trentadue: hep-ph/0402211

See e.g.: Table 1 of [1] and Figure 3.1 of [2] → Conclude:
The nonlogarithmic $O(\alpha^2)$ terms, originating from pure QED radiative 1-loop and from 2-loop diagrams are not completely covered.

They have to be calculated and integrated into the MC programs.

Beware:

$$m_e, m_\gamma, (d - 4), E_\gamma$$
**Results:** Numerical comparison in all $\bar{f} f$

**Bhabha $e^- e^+ \rightarrow e^- e^+ (\gamma)$ at LC:** $\sqrt{s} = 500$ GeV, $E_{\text{max}}(\gamma_{\text{soft}}) = \frac{\sqrt{s}}{10}$

<table>
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<th>[ \frac{d\sigma}{d\cos \theta} ]_{\text{Born}} (pb)</th>
<th>[ \frac{d\sigma}{d\cos \theta} ]_{Q(\alpha^3) = \text{Born+QED+weak+soft}}</th>
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Great independent agreement up to 14 digits! : limit in double precision

Previous agreement with *FeynArt*: 11 digits [hep-ph/0307132], SANC: 10 digits [hep-ph/0207156]

Thanks to T. Hahn, numbers supplied with *FeynArt* + *FormCalc* + *LoopTools*
Table 2:
The differential Bhabha cross section in nbarn as function of the scattering angle and the cms-energy.
\( M_Z = 91.16 \text{ GeV}, m_t = 150 \text{ GeV}, M_H = 100 \text{ GeV.} \)
Upper rows: \( DZ \), lower rows: \( H \).
\( \delta_m \): largest relative deviation in per mille.

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Bhabha scattering

Status 2005

**Know the constant term** \((m_e = 0)\)

**from 2-loop Bhabha scattering**

A. Penin, *Two-Loop Corrections to Bhabha Scattering*, hep-ph/0501120 v.3, \(\rightarrow\) PRL

Transform the massless 2-loop results of Bern, Dixon, Ghinculov (2002) with InfraRed (IR) regulation by \(D = 4 - 2\epsilon\) into the on-mass-shell renormalization with \(m_e \rightarrow 0\) and IR regulation by \(\lambda = m_\gamma \neq 0\)

Use IR-properties of amplitudes (see Penin):

[A ] *Exponentiation* of the IR logarithms (Sudakov 1956,...)

[B ] *Factorization* of the collinear logarithms into external legs (Frenkel, Taylor 1976)

[C ] *Non-renormalization* of the IR exponents (YFS 1961, ....)

Isolate the closed fermion loop contribution (does not fulfil [C]) and add it separately (Burgers 1985, Bonciani et al. 2005, Penin)

If all this is correct, the constant term in \(m_e\) is known for the MCs (but the radiative one-loops with 5-point functions).
Two Loop Bhabha Scattering

To calculate Bhabha scattering it is best to first compute $e^+e^- \rightarrow \mu^+\mu^-$, since it’s closely related but has less diagrams.

There are 47 QED diagrams contributing to $e^+e^- \rightarrow \mu^+\mu^-$. 

In this calculation all particles massless.

The Bhabha scattering amplitude can be obtained from $e^+e^- \rightarrow \mu^+\mu^-$ simply by summing it with the crossed amplitude (including fermi minus sign).
The massive 2-loop contributions

We are interested in a calculation of the virtual second order corrections to

$$\frac{d\sigma}{d \cos \vartheta}(e^+e^- \rightarrow e^+e^-)$$

We are using a scheme with

(1) $m_e \neq 0$ (good with the MC’s)
(2) $m_\gamma = 0$ (bad with the MC’s; $\rightarrow$ Mastroia, Remiddi 2003)
(3) dim.reg. for UV and IR divergences

Also:

Argeri, Bonciani, Ferroglia, Mastroia, Remiddi, v.d.Bij: all but many 2-boxes
Heinrich, Smirnov: Calculation of selected complicated Feynman integrals
There are few topologies only:

- 1-loop: 5
- 2-loop self energies: 5 (3 for external legs)
- 2-loop vertices: 5
- 2-loop boxes: 6 \(\rightarrow\) next slide

The many Feynman integrals may be reduced to 'few' master integrals by sophisticated methods (Remiddi-Laporta algorithm, 1996/2000 \(\rightarrow\) IdSolver (Czakon 2003)).

**The massive diagrams** (See also webpage)

Assume 3 leptonic flavors, do not look at loops in external legs.

Not too many QED diagrams:

- Born diagrams: 2
- 1-loop diagrams: 14
- 2-loop diagrams: 154 (with 68 double-boxes) interfere with Born
The two-loop box diagrams for massive Bhabha scattering

- **B5**: Completely known (2004)
  Czakon, Gluza, Riemann: [http://www-zeuthen.desy.de/.../MastersBhabha.m](http://www-zeuthen.desy.de/.../MastersBhabha.m) (unpubl.)

- **B1–B3**: Few masters known (Smirnov, Heinrich 2002, 2004)

- **B4, B6**: Not much known (Czakon et al. 2004)

The basic planar 2-box master of B1, B74m, was a breakthrough
The two-loop Feynman integrals

One has to solve many, very complicated Feynman integrals with \( L = 2 \) loops and \( N \leq 7 \) internal lines:

\[
G(X) = \frac{1}{(i\pi^{d/2})^2} \int \frac{d^D k_1 d^D k_2}{(q_1^2 - m_1^2)^{\nu_1} \cdots (q_j^2 - m_j^2)^{\nu_j} \cdots (q_N^2 - m_N^2)^{\nu_N}} X,
\]

\[X = 1, (k_1 P), (k_1 k_2), (k_2 P), \cdots\]

where \( P \) is some external momentum: \( p_1, \ldots p_4 \)

A completely numerical approach might be possible Passarino 2004.

For checks in the Euclidean region \((s < 0, t < 0)\) this has been proven to be a powerful tool Binoth, Heinrich 2000/03

We prefer to calculate the integrals analytically (where possible)

Derive a minimal set of so-called master integrals and algebraic expressions in terms of them for all the other Feynman integrals.
So we need a **A table of master integrals**

We use **IdSolver** with the Laporta/Remiddi algorithm:

Derive with integration-by-parts (and Lorentz-invariance) identities a system of *algebraic* equations for the Feynman integrals and solve the system.

- 1-loop: 5 masters (all known)
- 2-loop self energies: 6 masters (all known)
- 2-loop vertices: 19 masters (all known)
- 2-loop boxes: 33 masters → *(O(5) published, maybe more known)*

The calculation of the master integrals is mainly done with two methods:

- derive and solve (systems of) differential equations (with boundary conditions)
- derive and solve (up to 8-dimensional) *Mellin-Barnes integral representations* for
  single Feynman integrals
From Czakon et al., PRD 71 (2004): 4-point MIs entering basic two-loop box diagrams. An asterisk denotes one-loop MI. MIs with a dagger: know singular parts only

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<th>B1</th>
<th>B2</th>
<th>B3</th>
<th>B4</th>
<th>B5</th>
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A simple example: A class of scalar self-energy integrals

\[
\text{SE}312m(a,b,c,d) = -\frac{e^{2\epsilon \gamma_E}}{\pi^D} \int \frac{d^D k_1 d^D k_2 (k_1 k_2)^{-d}}{[(k_1 + k_2 - p)^2 - m^2]^b [k_1^2]^a [k_2^2 - m^2]^c}.
\]

The two Master Integrals are:

\[
\text{SE}312m = \text{SE}312m(1,1,1,0) \\
\text{SE}312md = \text{SE}312m(1,1,2,0)
\]

In Bonciani et al. 2003 it is used instead as a master integral:

\[
\text{SE}312mN = \text{SE}312m(1,1,1,-1)
\]

By an algebraic relation, valid for \( m^2 = 1 \) and \( p^2 = s \),

\[
\text{SE}312md = \frac{-(1+s) + \epsilon(2+s)}{s-4} \text{SE}312m + \frac{2(1-\epsilon)}{s-4} \left( T11m^2 + 3 \text{SE}312mN \right),
\]

one may derive then \text{SE}312md.

Because the integral \text{SE}312md is one of our masters, we reproduce it here explicitly:

\[
\text{SE}312md(x) = \frac{1}{2\epsilon^2} + \frac{1}{2\epsilon} - \left( \frac{1 - \xi_2}{2} + \frac{1+x}{1-x} H[0, x] + \frac{1+x^2}{(1-x)^2} H[0,0,x] \right) + \mathcal{O}(\epsilon).
\]

At our webpage, there is a file with all the master integrals we have determined so far.
The six two-loop 2-point MIs. External solid (dashed) lines describe on (off) -shell momenta.
The nine two-loop box MIs with seven internal lines.
The simplest diagram is the **tadpole**:

\[
T_{111m} = \frac{e^{\gamma_E}}{i\pi^{D/2}} \int \frac{d^D q}{q^2 - 1} = \frac{1}{\epsilon} + 1 + \left(1 + \frac{\zeta_2}{2}\right) \epsilon + \left(1 + \frac{\zeta_2}{2} - \frac{\zeta_3}{3}\right) \epsilon^2 + \ldots
\]
How to calculate 2-loop Bhabha masters?

- Self-energies and vertices and (very few) 2-boxes:
  use differential equations and Harmonic Polylogarithms, introduced by Remiddi, Vermaseren, plus ...)

- Some 7-line 2-boxes
  use Mellin-Barnes technique, sum up multiple series, use numerical checks in Euclidean space ($s,t$ negative)

- For the unsolved 2-boxes:
  Combination of both methods: present study

There are other methods not used here:

difference equations
pure numerical approaches

...
The 2-boxes with 5 lines

The completely known 2-boxes with 5 lines are B5I4m (Bonciani et al., Czakon et al. 2004), B5I2m1 (Czakon et al. 2004):

The divergent parts of the B5I2m2 and B5I2m3 type are known (Czakon et al. 2004):
**B5l3m**: The divergences in $D - 4 = -2\epsilon$

The **B5l3m** boxes contribute to **B2** (2nd planar 2-box) (shrink two lines...)

The **B5l3md2** topology appears twice as a master but the **B5l3md1** does not!
The B5l3m topology: Gross features

\[ MB5l3m[x, y] = Sum[B5l3m[k, x, y] \ast e^{p^k}, k, 0, 1]; \]  
\[ MB5l3md1[x, y] = Sum[B5l3md1[k, x, y] \ast e^{p^k}, k, -2, 1]; \]  
\[ MB5l3md2[x, y] = Sum[B5l3md2[k, x, y] \ast e^{p^k}, k, -2, 1]; \]  
\[ MB5l3md2a[x, y] = Sum[B5l3md2a[k, x, y] \ast e^{p^k}, k, -2, 1]; \]  
\[ MB5l3md3[x, y] = Sum[B5l3md3[k, x, y] \ast e^{p^k}, k, -1, 1]; \]

Note:
- B5l3m – the basic master is finite
- B5l3md2 – use 4-dim. MB-Representation
- B5l3md2’ – the same, but \((s \leftrightarrow t)\)
- B5l3md1, B5l3md3 – system of 2 coupled differential eqns
  Only BLB5l3md1 has \(1/\epsilon^2\) (so decouples), and last step is the two \(1/\epsilon\) coefficients of B5l3md1 and B5l3md3.
  The first one is found by algebraic manipulations (see Czakon et al. LCWS Paris 2004), the second then fulfils a diff.eqn
Differential equations

\[
\frac{\partial B5l3md3[-1]}{\partial x} = \frac{1 + x^2}{x(1 - x^2)} B5l3md3[-1] - \frac{yH[0, y]}{(1 - x^2)(1 - y^2)}
\]

(6)

with \( s = -(1 - x)^2 / x \), \( t = -(1 - y)^2 / y \)

Solution:

\[
B5l3md3[-1] = -\frac{xy}{(-1 + x^2)(-1 + y^2)} H[0, x] H[0, y]
\]

(7)

with

\[
H[0, x] = \ln(x)
\]

(8)

The coefficients in the equation are of the form

\[
\frac{A_1}{x - B_1} + \frac{A_2}{x - B_2} + ..
\]

(9)

One may derive (systems of ) differential equations for the masters, with inhomogeneity composed of simpler masters (Kotikov, Laporta, Remiddi)
\[
\frac{\partial M_n}{\partial x} = A(x, y) M_n + I(x, y)
\]  \hspace{1cm} (10)

\[
I(x, y) = \sum_{k=0, n-1} c_k M_k
\]  \hspace{1cm} (11)

Expand in \( \epsilon \) \((D = 4 - 2\epsilon)\):

\[
M_n = \sum_{i=-2, i_m} M_n, i \epsilon^i \text{ etc.}
\]  \hspace{1cm} (12)

General solution for homogeneous eqn. \((M_h' = A M_h)\):

\[
\frac{M_h'}{M_h} = A
\]  \hspace{1cm} (13)

\[
\int \left( \frac{M_h'}{M_h} \right) = \ln M_h = \int A
\]  \hspace{1cm} (14)

\[
= \int \sum \frac{a_i}{x - x_i} \sim \ln(x - x_i)
\]  \hspace{1cm} (15)

so:

\[
M_h \sim \text{Polynomials}
\]  \hspace{1cm} (16)

Then the inhomogeneous solution is:

\[
M(x, y) = M_h(x, y) \left( \text{const}(y) + \int \frac{I(x', y)}{M_h(x', y)} \right)
\]  \hspace{1cm} (17)
Result:
nested integrals over 'simple' iterated integrands

The method leads to the HPLs \( H(\{a\}, x) \) and GPLs \( G(\{a(y)\}, x) \)

Harmonic Polylogarithms \( H(x) \)

\[
H[-1, 1, x] = \int_0^x \frac{dx''}{(1 + x'')} \int_0^{x''} \frac{dx'}{(1 - x')} \quad (18)
\]

\[
= Li_2 \left( \frac{1 + x}{2} \right) + \ldots \quad (19)
\]

Generalized Harmonic Polylogarithms \( G(x, y) \) ...

but it works only if the polynomial structure is simple enough for a solution with this class of functions

Method is absolutely 'super' if it works.

But:
one needs complete chains of masters of lower complexity, and there are systems of up to 6 (!) potentially coupled 1st order equations
Mellin-Barnes representations


\[
\frac{1}{(A + B)^\nu} = \frac{B^{-\nu}}{(1 - (-A/B))^{-\nu}} = \frac{B^{-\nu}}{2\pi i \Gamma(\nu)} \int_{-i\infty}^{i\infty} d\sigma A^\sigma B^{-\sigma} \Gamma(-\sigma)\Gamma(\nu + \sigma) \tag{20}
\]

Is special case of a well-known Mellin-Barnes integral for hypergeometric functions

\[
\frac{1}{(1 - z)^\nu} = 2F_1(\nu, b, b', z)|_{b=b'} \tag{21}
\]

\[
= \frac{1}{2\pi i \Gamma(\nu) \Gamma(b')} \int_{-i\infty}^{+i\infty} d\sigma (-z)^\sigma \Gamma(\nu + \sigma)\Gamma(-\sigma) \frac{\Gamma(b + \sigma)}{\Gamma(b' + \sigma)} \tag{22}
\]

with \(-z = A/B\).

How can this be made useful here?
Introduce Feynman parameters

The momentum integrals of a Feynman diagram may be performed with Feynman parameters, one for each line. In 2-loops, consider two subsequent sub-loops (the first: off-shell 1-loop, second on-shell 1-loop) and get e.g. for B7l4m2, the planar 2nd type 2-box:
allow for propagators with indices, \(1/(k_1^2 - m_1^2)^{a_1}\) etc.

\[
K_{\text{1-loop Box, off}} = \frac{(-1)^4567 \Gamma(a_{4567} - d/2)}{\Gamma(a_4)\Gamma(a_5)\Gamma(a_6)\Gamma(a_7)} \int_0^\infty \prod_{j=4}^7 dx_j x_j^{a_j - 1} \delta(1 - x_4 - x_5 - x_6 - x_7) \frac{F^{a_{4567} - d/2}}{F_{a_{4567} - d/2}}
\]  

where \(a_{4567} = a_4 + a_5 + a_6 + a_7\) and the function \(F\) is characteristic of the diagram; here for the off-shell 1-box (2nd type):

\[
F = [-t]x_4x_7 + [-s]x_5x_6 + m^2(x_5 + x_6)^2
\]

\[
+ (m^2 - Q_1^2)x_7(x_4 + 2x_5 + x_6) + (m^2 - Q_2^2)x_7x_5
\]

We want to apply now:

\[
\int_0^1 \prod_{i}^4 dx_i x_i^{\alpha_i - 1} \delta(1 - x_1 - x_2 - x_3 - x_4) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\alpha_4)}{\Gamma(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)}
\]

with coefficients \(\alpha_i\) dependent on \(a_i\) and on \(F\).

For this, we have to apply several MB-integrals here.
And repeat the procedure for the 2nd subloop.
For the 2nd planar 2-box, B7\textbar{}4m2, one gets (Smirnov book 4.73):

\[
B_{\text{pl}, 2} = \frac{\text{const}}{(2\pi i)^6} \int_{-i\infty}^{+i\infty} \left[ \frac{m^2}{-s} \right]^{z_5 + z_6} \left[ \frac{-t}{-s} \right]^{z_1} \prod_{j=1}^6 [dz_j \Gamma(-z_j)] \frac{\prod_{k=7}^{18} \Gamma_k(\{z_i\})}{\prod_{l=19}^{24} \Gamma_l(\{z_i\})}
\]  

(27)

with \( a = a_1 + \ldots + a_7 \) and

\[
z_i = \text{const} + i \Im(z_i)
\]  

(28)

\[
d = 4 - 2\epsilon
\]  

(29)

\[
\text{const} = \frac{(i\pi^{d/2})^2 (-1)^a (-s)^{d-a}}{\Gamma(a_2)\Gamma(a_4)\Gamma(a_5)\Gamma(a_6)\Gamma(a_7)\Gamma(d - a_{4567})}
\]

(30)

The integrand includes e.g.:

\[
\Gamma_2 = \Gamma(-z_2)
\]  

(31)

\[
\Gamma_4 = \Gamma(-z_4)
\]  

(32)

\[
\Gamma_7 = \Gamma(a_4 + z_2 + z_4)
\]  

(33)

\[
\Gamma_8 = \Gamma(D - a_{445667} - z_2 - z_3 - 2z_4)
\]  

(34)

\[
\ldots
\]  

(35)
We now derive from B7\|4m2 the MB-integral for B5\|3m by setting $a_1 = 0$ (trivial, gives B6\|3m2) and $a_4 = 0$.

The latter do with care because of

$$\frac{1}{\Gamma(a_4)} \to \frac{1}{\Gamma(0)} = 0$$  \hspace{1cm} (36)

See by inspection that we will get factor $\Gamma(a_4)$ if $z_2, z_4 \to 0$.

$\to$ Start with the $z_2, z_4$ integrations by taking the residues for closing the integration contours to the right:

$$I_{2,4} = \frac{(-1)^2}{(2\pi i)^2} \int dz_2 \Gamma(-z_2) \int dz_4 \frac{\Gamma(a_4 + z_2 + z_4)}{\Gamma(a_4)} \Gamma(-z_4) R(z_i)$$  \hspace{1cm} (37)

$$= \frac{1}{(2\pi i)} \int dz_2 \Gamma(-z_2) \sum_{n=0,1,\ldots} \frac{(-1)^n}{n!} \frac{\Gamma(a_4 + z_2 + n)}{\Gamma(a_4)} R(z_i)$$  \hspace{1cm} (38)

$$= \sum_{n,m=0,1,\ldots} \frac{(-1)^{n+m}}{n!m!} \frac{\Gamma(a_4 + n + m)}{\Gamma(a_4)} R(z_i) \to_{a=0} 1$$  \hspace{1cm} (39)

So, setting $a_1 = a_4 = 0$ and eliminating $\int dz_2 dz_4$ with setting $z_2 = z_4 = 0$

we got a 4-fold Mellin-Barnes integral for B5\|3m

with $24 - 3 = 21$ $z_i$-dependent $\Gamma$-functions which may yield residua within four-fold sums.
As mentioned:

This formula has to be calculated now explicitely for the case

\[ B5l3md2 = \frac{B_2}{\epsilon^2} + \frac{B_1}{\epsilon} + B_0 \]  

(40)

\( B5l3md2 \) is a dotted master, with index \( a_2 = 2 \), all others are one

Next tasks:

- Find a region of definiteness of the n-fold MB-integral

  \[ \Re(z_1) = -1/80, \Re(z_3) = -33/40, \Re(z_5) = -21/20, \Re(z_6) = -59/160, \Re(\epsilon) = -1/10! \]  

  (41)

- Then go to the physical region where \( \epsilon << 1 \) by distorting the integration path step by step
  (adding each crossed residuum – per residue this means one integral less!!)

- Take integrals by sums over residua, i.e. introduce infinite sums

- Sum these infinite multiple series into some known functions of a given class, e.g. Nielsen polylogs, Harmonic polylogs or whatever is appropriate.
Here this means:

\[
\begin{align*}
B5l3md2 \quad &\Rightarrow \quad MB(4\text{-dim,fin}) + MB_3(3\text{-dim,fin}) \\
&+ \quad MB_{36}(2\text{-dim, } \epsilon^{-1}, \text{fin}) + MB_{365}(1\text{-dim, } \epsilon^{-2}, \epsilon^{-1,\text{fin}}) \\
&+ \quad MB_5(3\text{-dim,fin})
\end{align*}
\]

(42)

(43)

(44)

After these preparations e.g.:

\[
MB_{365}(1\text{-dim, } \epsilon^{-2}) \; \sim \; \frac{1}{\epsilon^2} \int dz_6 \frac{(-s)^{(z_6-1)} \Gamma(-z_6)^3 \Gamma(1 + z_6)}{8\Gamma(-2z_6)}
\]

(45)

\[
\sim \frac{1}{\epsilon^2} \sum_{n=0,\infty} \frac{(-1)^n (-s)^n \Gamma(1 + n)^3}{8n! \Gamma(-2(-1 - n))}
\]

(46)

\[
= - \frac{1}{\epsilon^2} \frac{ArcSin(\sqrt{s}/2)}{2\sqrt{4 - s^2}}
\]

(47)

\[
= \frac{1}{\epsilon^2} \frac{x}{4(1 - x^2)} H[0, x]
\]

(48)

Here were residua at \(z_6 = -n - 1, n = 0, 1, \ldots\) taken
The divergent parts of the masters B5l3m are:

\[
B5l3m[-2,x,-y] = B5l3m[-1,x,-y] = 0;
\]

\[
B5l3md1[-2,x,-y] = \frac{((-1 + x)^2 y((-1 + y^2 + 2 y H[0, y]))}{(8 x^2 (1 + y)^3)};
\]

\[
B5l3md1[-1,x,-y] = \frac{((y^2 (6 (-1 + x - x^2 + x^3) H[0, x] \nonumber
\quad - 6 (1 + x) (-2 - 2 x^2 + 2 y^2 + 2 x^2 y^2 +
\quad y z^2 - 2 x y z^2 + x^2 y z^2 + 2 (-2 x - y + 2 x y - x z^2 y - 2 x y z^2 +
\quad (-1 + x)^2 y H[-1, -y] + 3 (-1 + x)^2 y H[-1, y]) H[0, y] -
\quad 6 (-1 + x) H[0, -1, y] - 4 y H[0, 0, y] + 8 x y H[0, 0, y] - 4 x y H[0, 1, y] - 4 x y H[0, 1, y] +
\quad 2 x^2 y H[0, 1, y]))}{(24 x (1 + y) (1 + y)^3)});
\]

\[
B5l3md2[-2,x,-y] = -x/(1 - x^2)/4 H[0, x];
\]

\[
B5l3md2[-1,x,-y] = ((x^2 y H[0, x] H[0, y] -
\quad (-1 + y^2) (z^2 + 6 H[-1, 0, x] - 4 H[0, 0, x] - 2 x H[1, 0, x])))}{(4 (-1 + x^2) (-1 + y^2))};
\]

\[
B5l3md2a[-a,x,-y] = B5l3md2[-a,y,x], \quad a=-2,-1;
\]

\[
B5l3md3[-2,x,-y] = 0;
\]

\[
B5l3md3[-1,x,-y] = -((x y H[0, x] H[0, y]) /((-1 + x^2) (-1 + y^2)));
\]
Summary

A calculation of the constant 2-loop term for Bhabha scattering is derived from massless calculations Penin, Bonciani et al.

In parallel:

- A complete list of MASSIVE masters was derived (2004)
- Huge files with algebraic relations for all the reducible Feynman integrals needed for the interferences of boxes with Born (not complete, but fully understood)
- Essential progress for the massive 2-box master integral determination. Underway: Determination of all 2-box masters in a systematic approach use Generalized Harmonic Polylogarithms Remiddi, Vermaseren plus potentially . . .
- An unsolved problem is the systematic summation of the massive multiple sums after the MB-integral evaluation

It is also possible to do the massive 2-loop calculation with present computers.

Improve the existing MC-codes with that.

Care about the radiative 1-loops (with 5-point functions).