Introduction to Mellin-Barnes Representations
for Feynman Integrals

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- Introduction: 2-loop QED contributions to Bhabha scattering
- Barnes’ contour integrals for the hypergeometric function
- Loop momentum integrations with Feynman parameters for $L$-loop $n$-point functions
- Representation by Mellin-Barnes integrals
- Treatment of divergencies in $d = 4 - 2\epsilon$ (MB package)
- Numerical evaluations, nested infinite series, approximations, and all that
- Summary
Introduction: 2-loop QED contributions to Bhabha scattering

We are interested in a calculation of the virtual second order corrections to

\[
\frac{d\sigma}{d\cos \vartheta}(e^+e^- \rightarrow e^+e^-)
\]

We are using a scheme with

(1) \(m_e \neq 0\) (good with the MC’s)

(2) \(m_\gamma = 0\) (bad with the MC’s; → Mastrolia, Remiddi 2003)

(3) \text{dim.reg.} for UV and IR divergences

Also:

Argeri, Bonciani, Ferroglia, Mastrolia, Remiddi, v.d.Bij: all but many 2-boxes

G. Heinrich, V. Smirnov: Calculation of selected complicated Feynman integrals
There are few topologies only:

- 1-loop: 5
- 2-loop self energies: 5 (3 for external legs)
- 2-loop vertices: 5
- 2-loop boxes: 6 → next slides

The many Feynman integrals may be reduced to 'few' master integrals by sophisticated methods (Remiddi-Laporta algorithm, 1996/2000 → IdSolver (Czakon 2003)).

**The massive diagrams** (See also http://www-zeuthen.desy.de/theory/research/bhabha)

Assume 3 leptonic flavors, do not look at loops in external legs.

Not too many QED diagrams:

- Born diagrams: 2
- 1-loop diagrams: 14
- 2-loop diagrams: 154 (with 68 double-boxes) interfere with Born
Irreducible two-loop diagrams: 1/3
Irreducible two-loop diagrams: 2/3
Irreducible two-loop diagrams: 3/3

Diagram 49; topology 69

Diagram 50; topology 70

Diagram 51; topology 70

Diagram 52; topology 70

Diagram 53; topology 71

Diagram 54; topology 71

Diagram 55; topology 71

Diagram 56; topology 72

Diagram 57; topology 72

Diagram 58; topology 72

Diagram 59; topology 73

Diagram 60; topology 73

Diagram 61; topology 73

Diagram 62; topology 73

Diagram 63; topology 74

Diagram 64; topology 75

Diagram 65; topology 75

Diagram 66; topology 75

Diagram 67; topology 75

Diagram 68; topology 76
The two-loop box diagrams for massive Bhabha scattering

- **B5**: → 5-line masters + simpler, completely known (2004)
  Czakon, Gluza, Riemann: http://www-zeuthen.desy.de/.../MastersBhabha.m (unpubl.)

- **B1–B3**: → 7-line masters + simpler, few masters known (Smirnov, Heinrich 2002, 2004;
  for all planar masters the small mass limit: Czakon et al. 2006)

- **B4, B6**: → planar 6,5-line masters + simpler small mass limit known (Czakon et al. 2006)

The basic planar 2-box master of B1, B7l4m, was a breakthrough
The nine two-loop box MIs with seven internal lines.
The ten two-loop box MIs with six internal lines.
The 15 two-loop box MIs with five internal lines.
The $N_f > 1$ contributions

Actis, Czakon, Gluza, TR, under study

The eight additional master integrals with two different mass scales.

These 2-box-diagrams represent a three-scale problem: $s/m^2, t/m^2, M^2/m^2$
Barnes’ contour integrals for the hypergeometric function

Exact proof and further reading: Whittaker & Watson (CUP 1965) 14.5 - 14.52, pp. 286-290

Consider

\[ F(z) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} d\sigma (-z)^\sigma \frac{\Gamma(a + \sigma)\Gamma(b + \sigma)\Gamma(-\sigma)}{\Gamma(c + \sigma)} \]

where \(|\arg(-z)| < \pi\) (i.e. \((-z)\) is not on the neg. real axis) and the path is such that it separates the poles of \(\Gamma(a + \sigma)\Gamma(b + \sigma)\) from the poles of \(\Gamma(-\sigma)\).

1/\(\Gamma(c + \sigma)\) has no pole.

Assume \(a \neq -n\) and \(b \neq -n\), \(n = 0, 1, 2, \cdots\) so that the contour can be drawn.

The poles of \(\Gamma(\sigma)\) are at \(\sigma = -n, n = 1, 2, \cdots\), and it is:

\[ \text{Residue}[F[s] \text{ Gamma}[-s], \{s,n\}] = (-1)^n n! F(n) \]

Closing the path to the right gives then, by Cauchy’s theorem, for \(|z| < 1\) the
hypergeometric function $2F_1(a, b, c, z)$ (for proof see textbook):

$$\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} d\sigma(-z)^\sigma \frac{\Gamma(a + \sigma)\Gamma(b + \sigma)\Gamma(-\sigma)}{\Gamma(c + \sigma)} = \sum_{n=0}^{\infty} \frac{\Gamma(a + n)\Gamma(b + n)}{\Gamma(c + n)} \frac{z^n}{n!} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} 2F_1(a, b, c, z)$$

The continuation of the hypergeometric series for $|z| > 1$ is made using the intermediate formula

$$F(z) = \sum_{n=0}^{\infty} \frac{\Gamma(a + n)\Gamma(1 - c + a + n)\sin[(c - a - n)\pi]}{\Gamma(1 + n)\Gamma(1 - a + b + n)\cos(n\pi)\sin[(b - a - n)\pi]} (-z)^{-a-n}$$

$$+ \sum_{n=0}^{\infty} \frac{\Gamma(b + n)\Gamma(1 - c + b + n)\sin[(c - b - n)\pi]}{\Gamma(1 + n)\Gamma(1 - a + b + n)\cos(n\pi)\sin[(a - b - n)\pi]} (-z)^{-b-n}$$

and yields

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} 2F_1(a, b, c, z) = \frac{\Gamma(a)\Gamma(a - b)}{\Gamma(a - c)} (-z)^{-a} 2F_1(a, 1 - c + a, 1 - b + ac, z^{-1})$$

$$+ \frac{\Gamma(b)\Gamma(b - a)}{\Gamma(b - c)} (-z)^{-b} 2F_1(b, 1 - c + b, 1 - a + b, z^{-1})$$
Corollary I

Putting $b = c$, we see that

$$2F_1(a, b, b, z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n) z^n}{\Gamma(a) n!}$$

$$= \frac{1}{(1-z)^a} = \frac{1}{2\pi i \Gamma(a)} \int_{-i\infty}^{+i\infty} d\sigma \frac{(-z)^\sigma}{\Gamma(a+\sigma)\Gamma(-\sigma)}$$

This allows to replace sum by product:

$$\frac{1}{(A + B)^a} = \frac{B^{-a}}{(1 - (-A/B))^{-a}} = \frac{B^{-a}}{2\pi i \Gamma(a)} \int_{-i\infty}^{i\infty} d\sigma \frac{A^\sigma B^{-\sigma}}{\Gamma(a+\sigma)\Gamma(-\sigma)}$$
Barnes’ lemma

If the path of integration is curved so that the poles of $\Gamma(c - \sigma)\Gamma(d - \sigma)$ lie on the right of the path and the poles of $\Gamma(a + \sigma)\Gamma(b + \sigma)$ lie on the left, then

$$\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} d\sigma \Gamma(a + \sigma)\Gamma(b + \sigma)\Gamma(c - \sigma)\Gamma(d - \sigma) = \frac{\Gamma(a + c)\Gamma(a + d)\Gamma(b + c)\Gamma(b + d)}{\Gamma(a + b + c + d)}$$

It is supposed that $a, b, c, d$ are such that no pole of the first set coincides with any pole of the second set.

Scetchn of proof: Close contour by semicircle $C$ to the right of imaginary axis. The integral exists and $\int_C$ vanishes when $\Re(a + b + c + d - 1) < 0$. Take sum of residues of the integrand at poles of $\Gamma(c - \sigma)\Gamma(d - \sigma)$. The double sum leads to two hypergeometric functions, expressible by ratios of $\Gamma$-functions, this in turn by combinations of $\sin$, may be simplifies finally to the r.h.s.

Analytical continuation: The relation is proved when $\Re(a + b + c + d - 1) < 0$.

Both sides are analytical functions of e.g. $a$. So the relation remains true for all values of $a, b, c, d$ for which none of the poles of $\Gamma(a + \sigma)\Gamma(b + \sigma)$, as a function of $\sigma$, coincide with any of the poles of $\Gamma(c - \sigma)\Gamma(d - \sigma)$.

Corollary II Any real shift $k$: $\sigma + k, a - k, b - k, c + k, d + k$ together with $\int_{-k - i\infty}^{-k + i\infty}$ leaves the result true.
How can this be made useful in the context of Feynman integrals?

- Apply corollary I to propagators and get:

\[
\frac{1}{(p^2 - m^2)^a} = \frac{1}{2\pi i} \Gamma(a) \int_{-i\infty}^{i\infty} d\sigma \frac{(-m^2)^\sigma}{(p^2)^{a+\sigma}} \Gamma(a + \sigma) \Gamma(-\sigma)
\]

which may allow to perform the (massless) momentum integral (with index \(a\) of the line changed to \((a + \sigma)\)).

- Apply corollary I after introduction of Feynman parameters and after the momentum integration to the resulting \(F\)- and \(U\)-forms, in order to get a single monomial in the \(x_i\), which allows the integration over the \(x_i\):

\[
\frac{1}{[A(s)x_1^{a_1} + B(s)x_1^{b_1}x_2^{b_2}]^a} = \frac{1}{2\pi i} \Gamma(a) \int_{-i\infty}^{i\infty} d\sigma [A(s)x_1^{a_1}]^\sigma [B(s)x_1^{b_1}x_2^{b_2}]^{a+\sigma} \Gamma(a + \sigma) \Gamma(-\sigma)
\]

Both methods leave Mellin-Barnes (MB-) integrals to be performed afterwards.
A short remark on history

- **N. Usyukina, 1975**: "ON A REPRESENTATION FOR THREE POINT FUNCTION", Teor. Mat. Fiz. 22;
  a finite massless off-shell 3-point 1-loop function represented by 2-dimensional MB-integral

  N-point 1-loop functions represented by n-dimensional MB-integral

- **V. Smirnov, 1999**: "Analytical result for dimensionally regularized massless on-shell double box", Phys. Lett. B460 (1999);
  treat UV and IR divergencies by analytical continuation: shifting contours and taking residues 'in an appropriate way'

- **B. Tausk, 1999**: "Non-planar massless two-loop Feynman diagrams with four on-shell legs", Phys. Lett. B469 (1999);
  nice algorithmic approach to that, starting from search for some unphysical space-time dimension $d$ for which the MB-integral is finite and well-defined

  Tausk’s approach realized in Mathematica program MB.m, published and available for use
Loop momentum integrations with Feynman parameters for $L$-loop $n$-point functions

Consider an arbitrary $L$-loop integral $G(X)$ with loop momenta $k_l$, with $E$ external legs with momenta $p_e$, and with $N$ internal lines with masses $m_i$ and propagators $1/D_i$,

$$G(X) = \frac{1}{(i\pi^{d/2})^L} \int \frac{d^D k_1 \ldots d^D k_L}{D_1^{\nu_1} \ldots D_i^{\nu_i} \ldots D_N^{\nu_N}} X(k_1, \ldots, k_L).$$

$$D_i = q_i^2 - m_i^2 = \left[ \sum_{l=1}^{L} c_i^l k_l + \sum_{e=1}^{E} d_i^e p_e \right]^2 - m_i^2$$

The numerator may contain a tensor structure

$$X = (k_1^\alpha_1 \ldots k_L^\alpha_L) (p_{e_1}^\alpha_1 \ldots p_{e_L}^\alpha_L)$$

Some numerators are reducible, i.e. one may divide them out against the numerators a la:

$$\frac{2kp_e}{D_1[(k+p_e)^2 - m^2] \ldots D_N} \equiv \frac{[(k + p_e)^2 - m^2] - [k^2 - m_1^2] + (m^2 - m_e^2)}{D_1[(k+p_e)^2 - m^2] \ldots D_N}$$

$$= \frac{1}{D_1 \ldots D_N} - \frac{1}{[(k+p_e)^2 - m^2] \ldots D_N} + \frac{m^2 - m_e^2}{D_1[(k+p_e)^2 - m^2] \ldots D_N}$$
For a two-loop QED box diagram, it is e.g. $L = 2$, $E = 4$, and we have as potential simplest numerators: $k_1^2, k_2^2, k_1 k_2$ and $2E$ products $k_1 p_e, k_2 p_e$ compared to $N$ internal lines, $N = 5, 6, 7$. This gives $I = L + L(L - 1)/2 + 2E - N$ irreducible numerators of this type: $I(N) = 9 - N = 4, 3, 2$ here.

This observation is of practical importance: imagine you search for potential masters. Then you may take into the list of masters at most (here e.g.) $I(5) = 4$, or $I(6) = 3$, or $I(7) = 2$ such integrals.

Which momenta combinations are irreducible is partly dependent on the choice of momenta conventions (and fixed by that) and partly dependent on choice.

Message: When evaluating all F.I. by MB-integrals, one should consider numerator integrals, and it is not too complicated compared to scalar ones.

Now introduce Feynman parameters:

$$\frac{1}{D_1^{n_1} D_2^{n_2} \ldots D_N^{n_N}} = \frac{\Gamma(n_1 + \ldots + n_N)}{\Gamma(n_1) \ldots \Gamma(n_N)} \int_0^1 dx_1 \ldots \int_0^1 dx_N \frac{x_1^{n_1-1} \ldots x_N^{n_N-1} \delta(1 - x_1 \ldots - x_N)}{(x_1 D_1 + \ldots + x_N D_N)^{N_{\nu}}},$$

with $N_{\nu} = n_1 + \ldots + n_N$.

The denominator of $G$ contains, after introduction of Feynman parameters $x_i$, the momentum dependent function $m^2$ with index-exponent $N_{\nu}$:

$$(m^2)^{-(n_1+\ldots+n_N)} = (x_1 D_1 + \ldots + x_N D_N)^{-N_{\nu}} = (k_i M_{ij} k_j - 2Q_j k_j + J)^{-N_{\nu}}$$

Here $M$ is an $(L \times L)$-matrix, $Q = Q(x_i, p_e)$ an $L$-vector and $J = J(x_i x_j, m_i^2, p_e, p_{e1})$. 
The momentum integration is now simple:
Shift the momenta $k$ such that $m^2$ has no linear term in $\bar{k}$:

$$k = \bar{k} + (M^{-1})Q,$$

$$m^2 = \bar{k}M\bar{k} - QM^{-1}Q + J.$$

Remember:

$$M^{-1} = \frac{1}{(\det M)} \tilde{M},$$

where $\tilde{M}$ is the transposed matrix to $M$. The shift leaves the integral unchanged (rename $\bar{k} \to k$):

$$I_k(1) = \int \frac{Dk_1 \ldots Dk_L}{(kMk + J - QM^{-1}Q)^{N\nu}}.$$

Rotate now the $k^0 \to iK^0_E$ with $k^2 \to -k^2_E$ (and again rename $k^E \to k$):

$$I_k(1) \to (i)^L \int \frac{Dk^E_1 \ldots Dk^E_L}{(-k^EMk^E + J - QM^{-1}Q)^{N\nu}} = (-1)^{N\nu} (i)^L \int \frac{Dk_1 \ldots Dk_L}{[kMk - (J - QM^{-1}Q)]^{N\nu}}.$$

Call

$$\mu^2(x) = - (J - QM^{-1}Q)$$
and get

\[ I_k(1) = (-1)^{N_\nu (i)^L} \int \frac{Dk_1 \ldots Dk_L}{(kMk + \mu^2)^{N_\nu}}. \]

For 1-loop integrals - will use only those - we are ready. For L-loops go on and now diagonalize the matrix \( M \) by a rotation:

\[
\begin{align*}
    k \to k'(x) &= V(x) k, \\
    kMk &= k'M_{diag}k' \\
    \to \sum \alpha_i(x)k_i^2(x), \\
    M_{diag}(x) &= (V^{-1})^+MV^{-1} = (\alpha_1, \ldots, \alpha_L).
\end{align*}
\]

This leaves both the integration measure and the integral invariant:

\[ I_k(1) = (-1)^{N_\nu (i)^L} \int \frac{Dk_1 \ldots Dk_L}{(\sum \alpha_i k_i^2 + \mu^2)^{N_\nu}}. \]

Rescale now the \( k_i \),

\[ \bar{k}_i = \sqrt{\alpha_i} k_i, \]

with

\[
\begin{align*}
    d^Dk_i &= (\alpha_i)^{-D/2} d^D\bar{k}_i, \\
    \prod_{i=1}^{L} \alpha_i &= \det M,
\end{align*}
\]

and get the Euclidean integral to be calculated (and rename \( \bar{k} \to k \)):

\[ I_k(1) = (-1)^{N_\nu (i)^L} (\det M)^{-D/2} \int \frac{Dk_1 \ldots Dk_L}{(k_1^2 + \ldots + k_L^2 + \mu^2)^{N_\nu}}. \]
Use now (remembering that \( Dk = dk/(i\pi^{d/2}) \)):

\[
i^L \int \frac{Dk_1 \ldots Dk_L}{(k_1^2 + \ldots + k_L^2 + \mu^2)^{N'}} = \frac{\Gamma \left( N' \nu - \frac{D}{2} L \right)}{\Gamma \left( \nu \right)} \frac{1}{(\mu^2)^{N' - \nu \cdot D \cdot L/2}},
\]

\[
i^L \int \frac{Dk_1 \ldots Dk_L \, k_1^2}{(k_1^2 + \ldots + k_L^2 + \mu^2)^{N'}} = \frac{d}{2} \frac{\Gamma \left( N' \nu - \frac{D}{2} L - 1 \right)}{\Gamma \left( \nu \right)} \frac{1}{(\mu^2)^{N' - \nu \cdot D \cdot L/2 - 1}}.
\]

These formulae follow for \( L = 1 \) immediately from any textbook.

For \( L > 1 \), get it iteratively, with setting \((k_1^2 + k_2^2 + m^2)^N = (k_1^2 + M^2)^N, M^2 = k_2^2 + m^2, \) etc.

Finally, one gets:

\[
G(1) = (\det M)^{-D/2} \sum_{A_{ij}} A_{ij} x_i x_j - D + Q^2 \left( \rightarrow -J + Q^2 \right) \text{ for } L = 1.
\]

with

\[
U(x) = (\det M) \rightarrow 1 \text{ for } L = 1
\]

\[
F(x) = (\det M) \mu^2 = -J + Q^2 \left( \rightarrow -J + Q^2 \right) \text{ for } L = 1.
\]

\[
\sum A_{ij} x_i x_j.
\]
The vector integral differs by some numerator $k_i p_e$ and thus there is a single shift in the integrand, $k \to \bar{k} + U(x) \bar{M} Q$, the $\int d^d \bar{k} \bar{k}/(\bar{k}^2 + \mu^2) \to 0$, and no further changes:

$$G(k_1 \alpha) = (-1)^{N_\nu} \frac{\Gamma \left( N_\nu - \frac{D}{2} L \right)}{\Gamma(\nu_1) \cdots \Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j \ x_j^{\nu_j-1} \delta \left( 1 - \sum_{i=1}^N x_i \right) \frac{U(x)^{N_\nu - 1 - D(L+1)/2}}{F(x)^{N_\nu - DL/2}} \left[ \sum_l \tilde{M}_{1l} Q_l \right]_{\alpha},$$

Here also a tensor integral:

$$G(k_1 \alpha k_2 \beta) = (-1)^{N_\nu} \frac{\Gamma \left( N_\nu - \frac{D}{2} L \right)}{\Gamma(\nu_1) \cdots \Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j \ x_j^{\nu_j-1} \delta \left( 1 - \sum_{i=1}^N x_i \right) \frac{U(x)^{N_\nu - 2 - D(L+1)/2}}{F(x)^{N_\nu - DL/2}}$$

$$\times \sum_l \left[ [\tilde{M}_{1l} Q_l]_{\alpha} [\tilde{M}_{2l} Q_l]_{\beta} - \Gamma \left( N_\nu - \frac{D}{2} L - 1 \right) \frac{g_{\alpha\beta}}{2} U(x) F(x) \frac{(V_{1l}^{-1})^+ (V_{2l}^{-1})}{\alpha_l} \right].$$

The 1-loop case will be used in the following $L$ times for a sequential treatment of an $L$-loop integral (remember $\sum x_j D_j = k^2 - 2Qk + J$ and $F(x) = Q^2 - J$):

$$G([1, kp_e]) = (-1)^{N_\nu} \frac{\Gamma \left( N_\nu - \frac{D}{2} \right)}{\Gamma(\nu_1) \cdots \Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j \ x_j^{\nu_j-1} \delta \left( 1 - \sum_{i=1}^N x_i \right) \frac{[1, Q p_e]}{F(x)^{N_\nu - D/2}}$$
Come back to the evaluation of Bhabha boxes . . .

. . . and look at B7l4m2, the second planar double box:

Perform the momentum and Feynman parameter integrations first for the subloop with $k_1$, then for the second.

![Figure 1: The planar 6- and 7-line topologies.](image)
Integrating the Feynman parameters – get MB-Integrals

In 2-loops, consider two subsequent sub-loops (the first: off-shell 1-loop, second on-shell 1-loop) and get e.g. for B7l4m2, the planar 2nd type 2-box (for momenta see next page):

\[ K_{\text{1-loop Box, off}} = \frac{(-1)^{a_{4567}} \Gamma(a_{4567} - d/2)}{\Gamma(a_4)\Gamma(a_5)\Gamma(a_6)\Gamma(a_7)} \int_0^\infty \prod_{j=4}^7 dx_j x_j^{a_j-1} \frac{\delta(1 - x_4 - x_5 - x_6 - x_7)}{F^{a_{4567} - d/2}} \]

where \( a_{4567} = a_4 + a_5 + a_6 + a_7 \) and the function \( F \) is characteristic of the diagram; here for the off-shell 1-box (2nd type):

\[ F^{-(a_{4567} - d/2)} = \{-t\}[x_4x_7 + [-s]x_5x_6 + m^2(x_5 + x_6)^2 \]
\[ + (m^2 - Q_1^2)x_7(x_4 + 2x_5 + x_6) + (m^2 - Q_2^2)x_7x_5\}^{-(a_{4567} - d/2)} \]

We want to apply now:

\[ \int_0^1 \prod_{j=4}^7 dx_j x_j^{\alpha_j-1} \delta(1 - x_4 - x_5 - x_6 - x_7) = \frac{\Gamma(a_4)\Gamma(a_5)\Gamma(a_6)\Gamma(a_7)}{\Gamma(a_4 + \alpha_5 + \alpha_6 + \alpha_7)} \]

with coefficients \( \alpha_i \) dependent on \( a_i \) and on \( F \).

For this, we have to apply several MB-integrals here.

And repeat the procedure for the 2nd subloop.
On-shell example: B412m, the 1-loop on-shell box
but here use of another sequence of MB-integrals than in Smirnov’s book

\[
den = (x_4 d_4 + x_5 d_5 + x_6 d_6 + x_7 d_7) \text{ Expand} \bigg/ \text{ kinBha} \bigg/ m^2 \rightarrow 1 \text{ Expand}
\]

\[
Q = -\text{Coefficient}[den, k]/2 \text{ Simplify}
\quad = p_3 x_4 + p_2 x_5 - p_1 (x_4 + x_6)
\]

\[
M = \text{Coefficient}[den, k^2] \text{ Simplify}
\quad = x_4 + x_5 + x_6 + x_7 \rightarrow 1
\]

\[
J = \text{den Simplify}
\quad = t x_4
\]

\[
F[x] = (Q^2 - J M) \text{ Expand} \bigg/ \text{ kinBha} \bigg/ m^2 \rightarrow 1 \bigg/ u \rightarrow -s - t + 4 \text{ Expand}
\quad = (x_5 + x_6)^2 + (-s)x_5x_6 + (-t)x_4x_7
\]

\[
B412ma = mb[(x_5 + x_6)^2, -tx_7x_4 - sx_5x_6, nu, ga]
\]

\[
B412mb = B412ma \bigg/ (\text{ -sx_5x_6 - tx_4x_7})^{(-ga - nu)} \rightarrow
\quad \text{mb[(-s)x_5x_6, (-t)x_7x_4, nu+ga, de]}
\quad \bigg/ \text{((-s)x_5x_6)^de_} \rightarrow (-s)^{de} x_5^{de} x_6^{de}
\quad \bigg/ \text{((x_56^2)^ga \rightarrow (x_5 + x_6)^{(2ga)}}
\]
\[
\text{B4l2mc = B4l2mb /. (x5 + x6)^{(2ga)} ->}
\text{mb[x5, x6, -2ga, si]}
\text{/.((-t)x4x7)^si_ -> (-t)^si x4^si x7^si // ExpandAll}
\]
\[
= 1/(\Gamma[-2ga] \Gamma[\nu])
\text{inv2piI^3 (-s)^de (-t)^(-de - ga - nu)}
\text{x4^(-de - ga - nu) x5^(de + si) x6^(de + 2 ga - si) x7^(-de - ga - nu)}
\text{\Gamma[-de] \Gamma[-ga] \Gamma[de + ga + nu] \Gamma[-si] \Gamma[-2ga + si]}
\]
\[
\text{B4l2md = xfactor4[a4, x4, a5, x5, a6, x6, a7, x7] B4l2mc}
\]
\[
= .... (-s)^{de} (-t)^{-(de-ga-\nu)}
\text{x4^(-1+a4-de-ga-\nu) x5^(-1+a5+de+si) x6^(-1+a6+de+2ga-si) x7^(-1+a7-de-ga-\nu)}
\]
\[
\text{B4l2me =}
\text{B4l2md / .}
\text{x4^B4_ x5^B5_ x6^B6_ x7^B7_ -> xint4[x4^B4 x5^B5 x6^B6 x7^B7]}
\]
\[
= ...
\]
\[B4l2mf = B4l2me /.\]
\[\Gamma[a6 + de + 2 \text{ ga} - si] \Gamma[-si] \Gamma[a5 + de + si] \Gamma[-2 \text{ ga} + si]\]
\[\rightarrow \text{barne1}[si, a5 + de, -2 \text{ ga}, a6 + de + 2 \text{ ga}, 0]\]

This finishes the evaluation of the MB-representation for \(B4l2m\).
Some routines in mathematica which were used:

(* Barnes’ first lemma: \( \int d(\sigma_1) \Gamma(\sigma_1 p + \sigma_1) \Gamma(\sigma_2 p + \sigma_1) \Gamma(\sigma_1 m - \sigma_1) \Gamma(\sigma_2 m - \sigma_1) \) 
with \( 1/\text{inv2piI} = 2 \pi i \) *)

\[
\text{barne1}[\sigma_1, \sigma_1 p, \sigma_2 p, \sigma_1 m, \sigma_2 m] := \\
\frac{1}{\text{inv2piI}} \Gamma(\sigma_1 p + \sigma_1 m) \Gamma(\sigma_1 p + \sigma_2 m) \Gamma(\sigma_2 p + \sigma_1 m) \Gamma(\sigma_2 p + \sigma_2 m) / \Gamma(\sigma_1 p + \sigma_2 p + \sigma_1 m + \sigma_2 m)
\]

(* Mellin-Barnes integral: \((A+B)^{-\nu} = 1/(2 \pi i) \int d(\sigma) a^{\sigma} b^{-\nu - \sigma} \Gamma(-\sigma) \Gamma(\nu + \sigma) / \Gamma(\nu) \) *)

\[
\text{mb}[a_-, b_-, \nu_-, \sigma_] := \text{inv2piI} a^{\sigma} b^{-\nu - \sigma} \Gamma(-\sigma) \Gamma(\nu + \sigma) / \Gamma(\nu)
\]

(* After the k-integration, the integrand for \( \int \prod (d\tau \; \tau^{a_i - 1}) \delta(1 - \sum \tau_i) \) 
will be (L=1 loop) : xfactor \( F^{-\nu} Q(\tau).pe \) with \( \nu = a_1 + \ldots + a_n - d/2 \) *)

\[
\text{xfactor3}[a_1, x_1, a_2, x_2, a_3, x_3] := \\
\pi^{d/2} (-1)^{a_1 + a_2 + a_3} x_1^{a_1 - 1} x_2^{a_2 - 1} x_3^{a_3 - 1} \Gamma(a_1 + a_2 + a_3 - d/2) / \Gamma(a_1) \Gamma(a_2) \Gamma(a_3)
\]

(* xinti - the i-dimensional x - integration over Feynman parameters /16 06 2005 *)

\[
\text{xint3}[x_1^{a_1}, x_2^{a_2}, x_3^{a_3}] := \\
\Gamma(a_1 + 1) \Gamma(a_2 + 1) \Gamma(a_3 + 1) / \Gamma(a_1 + a_2 + a_3 + 3)
\]
Two different 2-dim. MB-representations for the massive 1-loop QED box:

(* s=(PP1+PP2)^2, t=(PP1-PP3)^2, s+t+u=4, scalar QED box *)
(* PRDfact = E^(ep EulerGamma) 1/(I Pi^(d/2)) *)
(* B4l2m = PRDfact * K4l2m[k-PP1, 0, k, 1, k-PP3, 0, k-PP1 - PP2, 1] *)
(* B4l2mINPUT with (\int dr1 dr3) is the on-shell QED box *)

B4l2mINPUT[s_, t_, m1_, m2_, m3_, m4_] = 
((-1)^(m1 + m2 + m3 + m4)*E^(ep EulerGamma)*inv2piI^2*Pi^(2 - d/2 - ep)*
(-s)^(2 - ep - m1 - m2 - m3 - m4 - r1 - r3)*(-t)^r3*Gamma[-r1]*
Gamma[4 - 2*ep - 2*m1 - m2 - 2*m3 - m4 - 2*r3]*Gamma[2 - ep - m1 - m2 - m3 - r1 - r3]*
Gamma[2 - ep - m1 - m3 - m4 - r1 - r3]*Gamma[-r3]*Gamma[m1 + r3]*Gamma[m3 + r3]*
Gamma[-2 + ep + m1 + m2 + m3 + m4 + r1 + r3])
/(Gamma[m1]*Gamma[m2]*Gamma[m3] * Gamma[4 - 2*ep - m1 - m2 - m3 - m4]*Gamma[m4]*
Gamma[4 - 2*ep - 2*m1 - m2 - 2*m3 - m4 - 2*r1 - 2*r3])

(* s=(PP1+PP2)^2, t=(PP1-PP3)^2, s+t+u=4, scalar QED box *)
(* PRDfact = E^(ep EulerGamma) 1/(I Pi^(d/2)) *)
(* B4l2m = PRDfact * K4l2m[k-PP1, 0, k, 1, k-PP3, 0, k-PP1 - PP2, 1] *)
(* B4l2mINPUT with (\int dr1 dr3) is the on-shell QED box *)
(* the MB-sequence deviates from e.g. Smirnov book, done for B5l2m3 *)

B4l2mINPUTvar[s_, t_, m1_, m2_, m3_, m4_] = ((-1)^(m1 + m2 + m3 + m4)*
E^(ep EulerGamma)*inv2piI^2*Pi^(2 - d/2 - ep)*
Gamma[-r1]*Gamma[m2 + r1]*
Gamma[m4 + r1]*Gamma[d/2 - m1 - m2 - m4 - r1 - r3]*
Gamma[d/2 - m2 - m3 - m4 - r1 - r3]*Gamma[-r3]*
Gamma[-d/2 + m1 + m2 + m3 + m4 + r1 + r3]*Gamma[m2 + m4 + 2*r1 + 2*r3])
/(Gamma[m1]*Gamma[m2]*Gamma[m3]*Gamma[d - m1 - m2 - m3 - m4]*Gamma[m4]*
Gamma[m2 + m4 + 2*r1])
The 2-dim. MB-representation for the 1-loop QED box with numerator $k.pe$ depends on the choice of momentum flow

$$B412mnumINPUTvar[pe_, s_, t_, m1_, m2_, m3_, m4_] = -((-1)^(m1 + m2 + m3 + m4)*E^{-ep*EulerGamma}*inv2piI^2*pe*(-s)^r1*(-t)^((d - 2*(m1 + m2 + m3 + m4 + r1 + r3))/2)*Gamma[-r1]*Gamma[-r3]*Gamma[-d/2 + m1 + m2 + m3 + m4 + r1 + r3]*(p1*Gamma[1 + m2 + r1]*Gamma[m4 + r1]*Gamma[m2 + m4 + 2*r1]*Gamma[1 + m2 + m4 + 2*r1 + 2*r3]*Gamma[(d - 2*(m1 + m2 + m4 + r1 + r3))/2]) + Gamma[m2 + r1]*((p1 - p3)*Gamma[m4 + r1]*Gamma[1 + m2 + m4 + 2*r1]*Gamma[m2 + m4 + 2*(r1 + r3)]*Gamma[(d - 2*(-1 + m1 + m2 + m4 + r1 + r3))/2] - p2*Gamma[1 + m4 + r1]*Gamma[m2 + m4 + 2*r1]*Gamma[1 + m2 + m4 + 2*r1 + 2*r3]*Gamma[(d - 2*(m1 + m2 + m4 + r1 + r3))/2]))/(Gamma[m1]*Gamma[m2]*Gamma[m3]*Gamma[1 + d - m1 - m2 - m3 - m4]*Gamma[m4]*Gamma[m2 + m4 + 2*r1]*Gamma[m2 + m4 + 2*r1 + 2*r3]))

A 3-dim. representation which is not derived by shrinking lines from 7-line box:

$$b512m2 = InputForm[((-1)^(a1 + a2 + a3 + a4 + a5)*E^{-2*ep*EulerGamma}*inv2piI^3*(-s)^((2 - a2 - a4 - a5 - ep - r1 - r3 + si) - r3*Gamma[-r1])*Gamma[2 - a2 - a4 - a5 - ep - r1 - r3]*Gamma[-3]*Gamma[a2 + r3]*Gamma[a4 + r3]*Gamma[4 - 2*a1 - a3 - 2*ep - si]*Gamma[-2 + a2 + a4 + a5 + ep + r1 + r3 - si]*Gamma[a1 + si]*Gamma[-2 + a1 + a3 + ep + si]*Gamma[4 - 2*a2 - 2*a4 - a5 - 2*ep - 2*r3 + si]*Gamma[2 - a2 - a4 - ep - r1 - r3 + si)] / (Gamma[a1]*Gamma[a2]*Gamma[a3]*Gamma[a4]*Gamma[a5])

Gamma[4 - a1 - a3 - 2*ep]*Gamma[4 - a2 - a4 - a5 - 2*ep + si]*Gamma[4 - 2*a2 - 2*a4 - a5 - 2*ep - 2*r1 - 2*r3 + si])
Another nice box with numerator, $B5l3m(p_\varepsilon \cdot k_1)$

We used it for the determination if the small mass expansion.

$$B5l3m(p_\varepsilon \cdot k_1) = \frac{m^{4\varepsilon} (-1)^{a_{12345}} e^{2\varepsilon \gamma E}}{\prod_{j=1}^5 \Gamma[a_i] \Gamma[5 - 2\varepsilon - a_{123}] (2\pi i)^4} \int_{-i\infty}^{+i\infty} d\alpha \int_{-i\infty}^{+i\infty} d\beta \int_{-i\infty}^{+i\infty} d\gamma \int_{-i\infty}^{+i\infty} d\delta \left[ (-s)^{(4 - 2\varepsilon - a_{12345}) - \alpha - \beta - \delta} (-t)^\delta \right]$$

$$\frac{\Gamma[-4 + 2\varepsilon + a_{12345} + \alpha + \beta + \delta]}{\Gamma[6 - 3\varepsilon - a_{12345} - \alpha]} \frac{\Gamma[-\alpha] \Gamma[-\beta]}{\Gamma[2 - \varepsilon - a_{13} - \alpha - \gamma]} \frac{\Gamma[6 - 3\varepsilon - a_{12345} - \alpha]}{\Gamma[7 - 3\varepsilon - a_{12345} - \alpha]} \frac{\Gamma[5 - 2\varepsilon - a_{123}] \Gamma[5 - 2\varepsilon - a_{123} - 2\alpha - \gamma]}{\Gamma[5 - 2\varepsilon - a_{1123} - 2\alpha - \gamma]} \frac{\Gamma[4 - 2\varepsilon - a_{1123} - 2\alpha - \gamma]}{\Gamma[5 - 2\varepsilon - a_{1123} - 2\alpha - \gamma]}$$

$$\left\{ (p_\varepsilon \cdot p_3) \Gamma[1 + a_4 + \delta] \Gamma[6 - 3\varepsilon - a_{1234}] - (p_\varepsilon \cdot p_1) \Gamma[7 - 3\varepsilon - a_{12345}] \Gamma[8 - 4\varepsilon - a_{11233445}] \Gamma[9 - 4\varepsilon - a_{112233445}] \right\}$$

$$\left[ \frac{\Gamma[2 - \varepsilon - a_{13} - \alpha - \gamma]}{\Gamma[3 - \varepsilon - a_{12} - \alpha]} \frac{\Gamma[5 - 2\varepsilon - a_{1123} - 2\alpha - \gamma]}{\Gamma[8 - 4\varepsilon - a_{11233445}] \Gamma[5 - 2\varepsilon - a_{1123} - 2\alpha - \gamma]} \frac{\Gamma[6 - 3\varepsilon - a_{12345} - \alpha]}{\Gamma[3 - \varepsilon - a_{12} - \alpha]} \frac{\Gamma[4 - 2\varepsilon - a_{1234} - \alpha - \beta - \delta]}{\Gamma[5 - 2\varepsilon - a_{123} - 2\alpha - \gamma]} \Gamma[4 - 2\varepsilon - a_{123} - 2\alpha - \gamma] \Gamma[5 - 2\varepsilon - a_{1123} - 2\alpha - \gamma] \right]$$

$$\left[ \frac{\Gamma[2 - \varepsilon - a_{12} - \alpha]}{\Gamma[3 - \varepsilon - a_{12} - \alpha]} \frac{\Gamma[5 - 2\varepsilon - a_{12} - \alpha]}{\Gamma[5 - 2\varepsilon - a_{12} - \alpha]} \frac{\Gamma[6 - 3\varepsilon - a_{12345} - \alpha]}{\Gamma[3 - \varepsilon - a_{12} - \alpha]} \frac{\Gamma[4 - 2\varepsilon - a_{1234} - \alpha - \beta - \delta]}{\Gamma[5 - 2\varepsilon - a_{12} - \alpha]} \frac{\Gamma[5 - 2\varepsilon - a_{12} - \alpha]}{\Gamma[5 - 2\varepsilon - a_{12} - \alpha]} \right]$$

$$\left\{ (p_\varepsilon \cdot p_3) \Gamma[1 + a_4 + \delta] \Gamma[6 - 3\varepsilon - a_{1234}] - (p_\varepsilon \cdot p_1) \Gamma[7 - 3\varepsilon - a_{12345}] \Gamma[8 - 4\varepsilon - a_{11233445}] \Gamma[9 - 4\varepsilon - a_{112233445}] \right\}$$
This kind of expression now has to be evaluated:

- Check special cases of indices, set lines to 1 (by setting \( a_i \to 0 \) if possible)
- Extract the \( \epsilon \)-dependence related to UV and IR singularities (see next pages)
- After that: may set \( s < 0, \, t < 0 \) and evaluate numerically Euclidean case
- Use sector decomposition for a numerical comparison - if you have a program for that
- Try to go Minkowskian in a numerical way (if you like this)
- Go on analytically, e.g. by taking residua \( \to \) get nested infinite sums from the residua
- Try to sum them up
**B4l2m2**

[Fleischer, Gluza, Lorca, TR 2006] B4l2m, the 1-loop QED box, with two photons in the s-channel; the Mellin-Barnes representation reads for finite $\epsilon$:

\[
B4l2m = \text{Box}(t, s) = \frac{e^{\epsilon \gamma_E}}{\Gamma[-2\epsilon](-t)^{(2+\epsilon)}} \frac{1}{(2\pi i)^2} \int_{-i\infty}^{+i\infty} dz_1 \int_{-i\infty}^{+i\infty} dz_2 \\
\frac{(-s)^{z_1} (m^2)^{z_2}}{(-t)^{z_1+z_2}} \Gamma[2 + \epsilon + z_1 + z_2] \Gamma^2[1 + z_1] \Gamma[-z_1] \Gamma[-z_2] \\
\Gamma^2[-1 - \epsilon - z_1 - z_2] \frac{\Gamma[-2 - 2\epsilon - 2z_1]}{\Gamma[-2 - 2\epsilon - 2z_1 - 2z_2]}
\]  

(7)

Mathematica package MB used for analytical expansion $\epsilon \rightarrow 0$: [Czakon:2005rk]
\[ B4l2m = -\frac{1}{\epsilon} I1 + \ln(-s) I1 + \epsilon \left( \frac{1}{2} \left[ \zeta(2) - \ln^2(-s) \right] I1 - 2I2 \right). \] (8)

with \( I1 \) being also the divergent part of the vertex function \( C_0(t; m, 0, m)/s = V3l2m/s \) (as is well-known):

\[ I1 = \frac{e^{\epsilon E}}{st} \frac{1}{2\pi i} \int_{-\frac{1}{2} - i\infty}^{-\frac{1}{2} + i\infty} d z_1 \left( \frac{m^2}{-t} \right)^{z_1} \frac{\Gamma[3] [-z_1] \Gamma[1 + z_1]}{\Gamma[-2z_1]} = \frac{1}{m^2 s} \frac{2y}{1 - y^2} \ln(y) \] (9)

with \( y = (\sqrt{1 - 4m^2/t} - 1)/(\sqrt{1 - 4m^2/t} + 1) \): close contour to left, take residua at \( (1 + z_1) = -n \), sum up with Mathematica:

\[
\text{Residue}[F[x]\text{Gamma}[1 + x], \{x, -n\}] \text{ // InputForm} = -(-1)^n F[-n]/n!
\]

\[
\text{Sum}[s^n \text{Gamma}[n + 1]^3/(n!\text{Gamma}[2 + 2n]), \{n, 0, \text{Infinity}\}] \text{ // InputForm} = (4 \text{ArcSin}[\text{Sqrt}[s]/2])/(\text{Sqrt}[4 - s] \text{Sqrt}[s])
\]

The \( I2 \) is more complicated:

\[
I2 = \frac{e^{\epsilon E}}{t^2} \frac{1}{(2\pi i)^2} \int_{-\frac{3}{4} - i\infty}^{-\frac{3}{4} + i\infty} d z_1 \left( \frac{-s}{-t} \right)^{z_1} \frac{\Gamma[-z_1] \Gamma[-2(1 + z_1)] \Gamma^2[1 + z_1]}{\Gamma[-2(1 + z_1 + z_2)] \Gamma[2 + z_1 + z_2]} \] (10)
The expansion of B4l2m at small $m^2$ and fixed value of $t$

With

$$m_t = \frac{-m^2}{t}, \quad (11)$$

$$r = \frac{s}{t}, \quad (12)$$

Look, under the integral, at $(-m^2/t)^{z_2}$, and close the path to the right. Seek the residua from the poles of $\Gamma$-functions with the smallest powers in $m^2$ and try to sum the resulting series. Automatize this, it is not too easy.

we have obtained a compact answer for $I_2$ with the additional aid of XSUMMER

$$[Moch:2005uc]$$

. The box contribution of order $\epsilon$ in this limit becomes:

$$B4l2m[t, s, m^2; +1] = \frac{1}{st} \left\{ 4\zeta_3 - 9\zeta_2 \ln(m_t) + \frac{2}{3} \ln^3(m_t) + 6\zeta_2 \ln(r) - \ln^2(m_t) \ln(r) \right.$$ \[13\]

$$+ \frac{1}{3} \ln^3(r) - 6\zeta_2 \ln(1 + r) + 2 \ln(-r) \ln(r) \ln(1 + r) - \ln^2(r) \ln(1 + r)$$

$$+ 2 \ln(r) \text{Li}_2(1 + r) + 2 \text{Li}_3(-r) \right\} + \mathcal{O}(m_t).$$
Shrinking of lines; seek the $\epsilon$-expansion

Go on with some study of the 2nd planar 2-box, B7l4m2 (see also Smirnov book 4.73):

$$B_{\text{pl}, 2} = \frac{\text{const}}{(2\pi i)^6} \int_{-i\infty}^{+i\infty} \left[ \frac{m^2}{s} \right]^{z_5 + z_6} \left[ \frac{-t}{-s} \right]^{z_1} \prod_{j=1}^{6} [dz_j \Gamma(-z_j)] \frac{\prod_{k=7}^{18} \Gamma_k(\{z_i\})}{\prod_{l=19}^{24} \Gamma_l(\{z_i\})}$$

with $a = a_1 + \ldots + a_7$ and

$$z_i = \text{const} + i \Im(z_i)$$

$$d = 4 - 2\epsilon$$

$$\text{const} = \frac{(i\pi^{d/2})^2 (-1)^a (-s)^{d-a}}{\Gamma(a_2)\Gamma(a_4)\Gamma(a_5)\Gamma(a_6)\Gamma(a_7)\Gamma(d - a_{4567})}$$

The integrand includes e.g.:

$$\Gamma_2 = \Gamma(-z_2)$$
$$\Gamma_4 = \Gamma(-z_4)$$
$$\Gamma_7 = \Gamma(a_4 + z_2 + z_4)$$
$$\Gamma_8 = \Gamma(D - a_{445667} - z_2 - z_3 - 2z_4)$$

...
Figure 2: The planar 6- and 7-line topologies.

Figure 3: The 5-line topologies. B7l4m2: shrink line 1 get B6l3m2, then line 4 get B5l3m
Example:
derive from B7l4m2 the MB-integral for B5l3m by setting \( a_1 = 0 \) (trivial, gives B6l3m2) and then setting \( a_4 = 0 \).
The latter do with care because of
\[
\frac{1}{\Gamma(a_4)} \rightarrow \frac{1}{\Gamma(0)} = 0
\]
See by inspection that we will get factor \( \Gamma(a_4) \) if \( z_2, z_4 \rightarrow 0 \).

→ Start with the \( z_2, z_4 \) integrations by
taking the residues for closing the integration contours to the right:
\[
I_{2,4} = \frac{(-1)^2}{(2\pi i)^2} \int dz_2 \Gamma(-z_2) \int d\zeta_4 \frac{\Gamma(a_4 + z_2 + z_4)}{\Gamma(a_4)} \Gamma(-z_4) R(z_i)
\]
\[
= \frac{1}{(2\pi i)} \int d\zeta_2 \Gamma(-\zeta_2) \sum_{n=0,1,\ldots} \frac{(-1)^n}{n!} \frac{\Gamma(a_4 + z_2 + n)}{\Gamma(a_4)} R(z_i)
\]
\[
= \sum_{n,m=0,1,\ldots} \frac{(-1)^{n+m}}{n! m!} \frac{\Gamma(a_4 + n + m)}{\Gamma(a_4)} R(z_i) \rightarrow a_4 = 0 \times R(z_i)
\]
So, setting \( a_1 = a_4 = 0 \) and eliminating \( \int dz_2 d\zeta_4 \) with setting \( z_2 = z_4 = 0 \)
we got a 4-fold Mellin-Barnes integral for topology B5l3m (by "shrinking of lines")
with \( 24 - 3 = 21 \) \( z_i \)-dependent \( \Gamma \)-functions which may yield residua within four-fold sums.
The MB-representation has to be calculated explicitly at fixed indices, e.g.

\[ B_{5l3md2} = \frac{B_2}{\epsilon^2} + \frac{B_1}{\epsilon} + B_0 \]

**General Tasks, first two steps automated by MB.m:**

- Find a region of definiteness of the n-fold MB-integral

  \[ \Re(z_1) = -1/80, \Re(z_3) = -33/40, \Re(z_5) = -21/20, \Re(z_6) = -59/160, \Re(\epsilon) = -1/10! \]

- Then go to the physical region where \( \epsilon \ll 1 \) by distorting the integration path step by step (adding each crossed residuum – *per residue this means one integral less!!!*)

- Take integrals by sums over residua, i.e. introduce infinite sums

- Sum these infinite multiple series into some known functions of a given class, e.g. Nielsen polylogs, Harmonic polylogs or whatever is appropriate.
An important tool is the command FindInstance of Mathematica 5: It allows to solve a system of inequalities. Here an example for B7l4m3, the non-planar massive double box:

```
sol = FindInstance[
    Cases[B7l4m3 ... Gamma[x_] -> x > 0 /. ep -> -1/10, {z1, z2, z3, z4, z5, z6, z7, z8}]
```

The result is:

```
{z1 -> -1/20, z2 -> -1/40, z3 -> -1/20, z4 -> -29/32, 
z5 -> -67/80, z6 -> -83/160, z7 -> -273/320, z8 -> -5/64}
```

Really, all arguments are positive:

```
```

Now set \( \epsilon = 0 \):

```
{z1 -> -1/20, z2 -> -1/40, z3 -> -1/20, z4 -> -29/32, 
z5 -> -67/80, z6 -> -83/160, z7 -> -273/320, z8 -> -5/64, \ ep -> 0}
```

Determine again the arguments of the Gamma-functions; observe:

2 arguments are negative now: those for G3 and G8

```
```

Perform the corresponding shifts of integration curve, add the residua and again perform the test for the arguments of the new, lower-dimensional MB-integrals.
We derived an algorithmic solution for isolating the singularities in $1/\epsilon$.
The automatization of that: MB.m (M. Czakon)

\[
B5l3md2 \rightarrow MB(4\text{-dim, fin}) + MB_3(3\text{-dim, fin}) \\
+ MB_{36}(2\text{-dim, } \epsilon^{-1}, \text{fin}) + MB_{365}(1\text{-dim, } \epsilon^{-2}, \epsilon^{-1}, \text{fin}) \\
+ MB_5(3\text{-dim, fin})
\]

After these preparations e.g.:

\[
MB_{365}(1\text{-dim, } \epsilon^{-2}) \sim \frac{1}{\epsilon^2} \frac{1}{2\pi i} \int dz_6 \frac{(-s)^{(-z_6-1)} \Gamma(-z_6)^3 \Gamma(1+z_6)}{8\Gamma(-2z_6)} \\
= \frac{1}{\epsilon^2} \sum_{n=0,\infty} \frac{(-1)^n(-s)^n \Gamma(1+n)^3}{8n!\Gamma(-2(-1-n))} \\
= -\frac{1}{\epsilon^2} \frac{ArcSin(\sqrt{s}/2)}{2\sqrt{4-s\sqrt{s}}} \\
= \frac{1}{\epsilon^2} \frac{-x}{4(1-x^2)} H[0,x]
\]

Here residua were taken at $z_6 = -n - 1, n = 0, 1, \ldots$, and $H[0,x] = \ln(x)$ and $x = \frac{\sqrt{-s+4}-\sqrt{-s}}{\sqrt{-s+4}+\sqrt{-s}}$. T. Riemann, 23 July 2006 - CALC, Dubna
Summary

- We have introduced to the representation of $L$-loop $N$-point Feynman integrals of general type.
- The determination of the $\epsilon$-poles is generally solved.
- The remaining problem is the evaluation of the multi-dimensional, finite MB-Integrals.
- This is unsolved in the general case, so you have something to do if you like to . . .

Problem: Determine the small mass limit of $B5l2m2$ or of any other of the 2-loop boxes for Bhabha scattering. Stefano Actis may check your solution. He leaves soon.
Some facts on $\Gamma(z)$ and $\Psi(z)$

Residues may be easily derived from Laurent series expansions obtained with Mathematica, e.g.

\begin{align*}
\text{Series}[\text{Gamma}[a - z]^2, z, a, 1] &= \frac{1}{(z - a)^2} + \frac{2\gamma_E}{z - a} + 2\gamma_E^2 + \zeta_2 + \ldots \quad (14) \\
\text{Series}[\text{Gamma}[z - a]^2, z, a, 1] &= \frac{1}{(z - a)^2} - \frac{2\gamma_E}{z - a} + 2\gamma_E^2 + \zeta_2 + \ldots \quad (15) \\
\text{Series}[\text{Gamma}[a - z], z, a, 1] &= -\frac{1}{(z - a)} - \gamma_E + \ldots \quad (16) \\
\text{Series}[\text{Gamma}[z - a], z, a, 1] &= +\frac{1}{(z - a)} - \gamma_E + \ldots
\end{align*}

The following property of $\Gamma$ is useful:

$$
\Gamma(-n + \epsilon) = \frac{(-1)^n}{n!} \left[ \frac{1}{\epsilon} + \Psi(n + 1) + \ldots \right]
$$

The derivation: Express $\Gamma(-n + \epsilon)$ by $\Gamma(\epsilon)$. Thereby factors $1/(-n + \epsilon + k)$ are collected, whose inversion for small $\epsilon$ leads to the $\Psi$ term and the factor $(-1)^n/n!$. Potential terms with
\( \gamma_E \) cancel in order \( \epsilon^0 \). Have in mind here that:

\[
\text{Polygamma}[n + 1] = \text{Polygamma}[0, n + 1] \\
= \Psi(n + 1) \\
= \frac{\Gamma'(n + 1)}{\Gamma(n + 1)} \\
= S_1(n) - \gamma_E \\
= \sum_{k=1}^{n} \frac{1}{k} - \gamma_E
\]  

The following properties hold:

\[
\Psi(z + 1) = \Psi(z) + 1/z \\
\Psi(1 + \epsilon) = -\gamma_E + \zeta_2 \epsilon + \ldots \\
\Psi(1) = -\gamma_E \\
\Psi(2) = 1 - \gamma_E \\
\Psi(3) = 3/2 - \gamma_E
\]
Some facts on residua

The function

\[ F(z) = \sum_{i=-N}^{\infty} \frac{a_i}{(z-z_0)^i} \]

has the residue

\[ \text{Res } F(z) \big|_{z=z_0} = a_{-1} \]

An integral over an anti-clockwise directed closed path \( C \) around \( z_0 \) then is

\[ \frac{1}{2\pi i} \int_C dz F(z) = 2\pi i a_1 \]

If \( G(z) \) has a Taylor expansion around \( z_0 \) and \( F(z) \) has a Laurent expansion beginning with \( a_{-N}/(z-z_0)^N + \ldots \), then their product has the residue:

\[ \text{Res}[G(z) F(z)] \big|_{z=z_0} = \sum_{k=1}^{N} \frac{a_{-k} G(z_0)^{(k)}}{k!} \]
Some sums Mathematica can do

\[
\text{Sum}[s^n \Gamma(n + 1)^3/(n! \Gamma(2 + 2n)), n, 0, \infty] = \\
\frac{4 \arcsin\left(\sqrt{s}/2\right)}{\sqrt{4 - s} \sqrt{s}}
\]

\[
\text{Sum}[s^n \text{PolyGamma}[0, n + 1], n, 0, \infty] = \\
\frac{\text{EulerGamma} + \log(1 - s)}{-1 + s}
\]