Algebraic tensor Feynman integral reduction

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Corfu Summer Institute, Aug 29 – Sep 5, 2010, Corfu, Greece

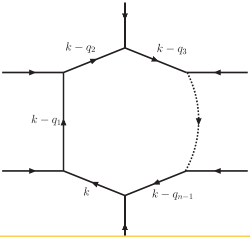
Introduction

1-loop *n*-point tensor integrals of rank *R*: (n,R)-integrals

$$I_n^{\mu_1\cdots\mu_R} = \int \frac{d^dk}{i\pi^{d/2}} \frac{\prod_{r=1}^{R} k^{\mu_r}}{\prod_{j=1}^{n} c_j^{\nu_j}}, \qquad (1)$$

$d = 4 - 2\epsilon$ and denominators c_j have *indices* ν_j and *chords* q_j

$$c_j = (k - q_j)^2 - m_j^2 + i\varepsilon \qquad (2)$$



tensor integrals due to:

- fermion propagators
- three-gauge boson couplings
- e.g. unitary gauge propagators

A simple example

1-loop self-energy:

$$I_2^\mu = \int \frac{d^dk}{i\pi^{d/2}} \frac{k^\mu}{[k^2 - M_1^2]\,[(k+p)^2 - M_2^2]} = p_\mu B_1$$

Solve:

$$\begin{aligned}
p_\mu I_2^\mu &= p^2 B_1(p, M_1, M_2) \\
&= \int \frac{d^dk}{i\pi^{d/2}} \frac{pk}{[k^2 - M_1^2]\,[(k+p)^2 - M_2^2]} = \int \frac{d^dk}{i\pi^{d/2}} \frac{pk}{D_1\ D_2} \\
&= \int \frac{d^dk}{i\pi^{d/2}} \left[\frac{D_2 - (p^2 - M_2^2 - M_1^2) - D_1}{D_1\ D_2}\right], \\
B_1(p, M_1, M_2) &= \frac{1}{2p^2}\left[A_0(M_1) - A_0(M_2) - (p^2 - M_2^2 - M_1^2) B_0(p, M_1, M_2)\right]
\end{aligned}$$

A tensor Feynman integral is expressed in terms of scalar Feynman integrals.

Systematic approach:
Passarino, Veltman 1978 [1]

Need in addition a library of scalar functions:
'tHooft, Veltman 1979 [2]

State of the art:
Hahn, LoopTools/FF [3, 4]

This talk: derive efficient reduction formulae in the algebraic Fleischer-Davydychev-Tarasov approach

The original Passarino-Veltman reduction allows to express tensor integrals by a small set of scalar 4-,3-,2-,1-point functions integrals in d dimensions.

- Need Extensions: Reduction of n-point functions with $n > 4$
- Need Improvements: Avoid the break-down in certain kinematical configurations

Recent developments in the Fleischer-Davydychev-Tarasov approach

- get tensor reduction such that one may $\cdots$:
- $\cdots$ kill pentagon Gram determinants
- $\cdots$ treat sub-diagram Gram determinants

Outline

[5] 1991 Davydychev, ... *Reducing Feynman diagrams to scalar integrals*
[6] 1996 Tarasov, *Connection [of] Feynman integrals [with] different ... space-time dimensions*
[7] 1999 Fleischer et al., *Algebraic reduction of one-loop Feynman graph amplitudes*

References:
[8] Diakonidis et al., PRD 80 (2009) 036003
[9] Diakonidis et al., PLB 683 (2010) 69
[10] J. Fleischer, T. Riemann, PoS(ACAT2010)074 [arXiv:1006.0679] and unpubl. work

Notations: $I_{n-1}^{\{\mu_1,\cdots\},s}$ etc.

The tensor integral $I_{n-1,ab}^{\{\mu_1,\cdots\},s}$ is obtained from the integral $I_n^{\{\mu_1,\cdots\}}$ by

- shrinking line s
- raising the powers of inverse propagators a, b

$$\mathbf{s}^- \, \mathbf{a}^+ \, \mathbf{b}^+ \;\; I_n^{\{\mu_1,\cdots\}} \;\; = \;\; I_{n-1,ab}^{\{\mu_1,\cdots\},s} \qquad (3)$$

The operators $\mathbf{i}^\pm, \mathbf{j}^\pm, \mathbf{k}^\pm$
act by shifting the indices ν_i, ν_j, ν_k by ± 1.

Notations: Gram and modified Cayley determinant, signed minors [Melrose:1965]

Gram determinant G_n:

$$G_n = |2q_iq_j|, i,j = 1,\dots n \qquad (4)$$

Modified Cayley determinant $()_N$ of a diagram with N internal lines and chords q_j:

$$()_N \equiv \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & Y_{11} & Y_{12} & \dots & Y_{1N} \\ 1 & Y_{12} & Y_{22} & \dots & Y_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1N} & Y_{2N} & \dots & Y_{NN} \end{vmatrix}, \qquad (5)$$

with matrix elements

$$Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2, \quad (i,j = 1\dots N) \qquad (6)$$

For a choice $q_n = 0$, both determinants are related: $()_N = -G_{N-1}$

$\Rightarrow$ The Gram determinant $()_N$ does not depend on the masses.

Notations: signed minors [Melrose:1965]

signed minors of $()_N$ are constructed by deleting m rows and m columns from $()_N$, and multiplying with a sign factor:

$$\left(\begin{array}{cccc} j_1 & j_2 & \cdots & j_m \\ k_1 & k_2 & \cdots & k_m \end{array}\right)_N \equiv$$
$$\equiv (-1)^{\sum_l (j_l+k_l)} \mathrm{sgn}_{\{j\}}\, \mathrm{sgn}_{\{k\}} \left| \begin{array}{c} \text{rows } j_1 \cdots j_m \text{ deleted} \\ \text{columns } k_1 \cdots k_m \text{ deleted} \end{array} \right| \tag{7}$$

where $\mathrm{sgn}_{\{j\}}$ and $\mathrm{sgn}_{\{k\}}$ are the signs of permutations that sort the deleted rows $j_1 \cdots j_m$ and columns $k_1 \cdots k_m$ into ascending order.

Example:

$$\left(\begin{array}{c} 0 \\ 0 \end{array}\right)_N \equiv \left| \begin{array}{cccc} Y_{11} & Y_{12} & \ldots & Y_{1N} \\ Y_{12} & Y_{22} & \ldots & Y_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1N} & Y_{2N} & \ldots & Y_{NN} \end{array} \right|, \tag{8}$$

Example: Getting a 4-point function from a six-point function I

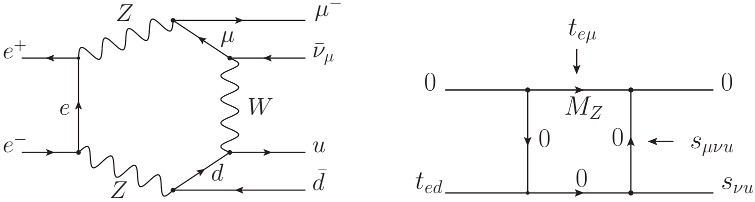


Figure: A six-point topology (a) leading to four-point functions (b) with realistically vanishing Gram determinants.

Example: Getting a 4-point function from a six-point function II

The example is taken from [11].

The corresponding 4-point tensor integrals are, in LoopTools [3, 12] notation:

$$\mathrm{D0i}(\mathrm{id}, 0, 0, s_{\bar{\nu}u}, t_{ed}, t_{\bar{e}\mu}, s_{\mu\bar{\nu}u}, 0, M_Z^2, 0, 0). \tag{9}$$

The Gram determinant is:

$$()_4 \quad = \quad -2t_{\bar{e}\mu}[s^2_{\mu\bar{\nu}u} + s_{\bar{\nu}u}t_{ed} - s_{\mu\bar{\nu}u}(s_{\bar{\nu}u} + t_{ed} - t_{\bar{e}\mu})], \tag{10}$$

It vanishes if:

$$t_{ed} \to t_{ed,\mathrm{crit}} = \frac{s_{\mu\bar{\nu}u}(s_{\mu\bar{\nu}u} - s_{\bar{\nu}u} + t_{\bar{e}\mu})}{s_{\mu\bar{\nu}u} - s_{\bar{\nu}u}}. \tag{11}$$

In terms of a dimensionless scaling parameter x,

$$t_{ed} \quad = \quad (1 + x)t_{ed,\mathrm{crit}}, \tag{12}$$

Example: Getting a 4-point function from a six-point function III

the Gram determinant becomes:

$$()_4 = 2\, x\, s_{\mu\bar{\nu}u} t_{\bar{e}\mu} (s_{\mu\bar{\nu}u} - s_{\bar{\nu}u} + t_{\bar{e}\mu}). \tag{13}$$

We will also need the modified Cayley determinant:

$$\begin{aligned}
\binom{0}{0}_4 &= \begin{pmatrix} 2M_Z^2 & M_Z^2 & M_Z^2 - s_{\mu\bar{\nu}u} & M_Z^2 \\ M_Z^2 & 0 & -s_{\bar{\nu}u} & M_Z^2 \\ M_Z^2 - s_{\mu\bar{\nu}u} & -s_{\bar{\nu}u} & 0 & -t_{ed} \\ M_Z^2 & -t_{\bar{e}\mu} & -t_{ed} & 0 \end{pmatrix} \\
&= s_{\mu\bar{\nu}u}^2 t_{\bar{e}\mu}^2 + 2\, M_Z^2 t_{\bar{e}\mu} [-2 s_{\bar{\nu}u} t_{ed} + s_{\mu\bar{\nu}u}(s_{\bar{\nu}u} + t_{ed} - t_{\bar{e}\mu})] \\
&\quad + M_Z^4 (s_{\bar{\nu}u}^2 + (t_{ed} - t_{\bar{e}\mu})^2 - 2 s_{\bar{\nu}u}(t_{ed} + t_{\bar{e}\mu})).
\end{aligned}$$

Recursions for hexagons

Express any hexagon by pentagons

[Fleischer:1999,Binoth:2005,Denner:2005,Diakonidis:2008 [7, 13, 14, 8]]

$$I_6^{\mu_1\cdots\mu_{R-1}\rho} = -\sum_{s=1}^{6} I_5^{\mu_1\cdots\mu_{R-1},s}\,\bar{Q}_s^{\rho}. \tag{14}$$

auxiliary vectors

$$\bar{Q}_s^{\rho} \quad = \quad \sum_{i=1}^{6} q_i^{\rho}\,\frac{\binom{0s}{0i}_6}{\binom{0}{0}_6} \quad , \quad s = 1\ldots 6. \tag{15}$$

Dimensional shifts and recurrence relations for pentagons (I)

Following [Davydychev:1991 [5]]

Replace tensors by scalar integrals in higher dimensions:

Example $R = 3$:

$$I_5^{\mu\nu\lambda} = \int \frac{d^{4-2\epsilon}k}{i\pi^{d/2}} \prod_{r=1}^{5} c_r^{-1} k^\mu k^\nu k^\lambda \tag{16}$$

$$= - \sum_{i,j,k=1}^{4} q_i^\mu q_j^\nu q_k^\lambda n_{ijk} I_{5,ijk}^{[d+]^3} + \frac{1}{2} \sum_{i=1}^{n-1} (g^{\mu\nu} q_i^\lambda + g^{\mu\lambda} q_i^\nu + g^{\nu\lambda} q_i^\mu) I_{5,i}^{[d+]^2},$$

and $n_{ijk} = (1 + \delta_{ij})(1 + \delta_{ik} + \delta_{jk})$.

$[d+]^l = 4 - 2\epsilon + 2l$, and for definition of $I_{5,i}^{[d+]^2}$ etc. see (3).

Dimensional shifts and recurrence relations for pentagons (II)

'Naive', direct approach – just perform dimensional recurrences

Following [Tarasov:1996,Fleischer:1999 [6, 7]]
apply recurrence relations, relating scalar integrals of different dimensions, in order to get rid of the dimensionalities $[d+]^l = 4 - 2\epsilon + 2l$:

$$\nu_j (\mathbf{j}^+ I_5^{[d+]}) = \frac{1}{()_5} \left[-\binom{j}{0}_5 + \sum_{k=1}^{5} \binom{j}{k}_5 \mathbf{k}^- \right] I_5 \tag{17}$$

$$(d - \sum_{i=1}^{5} \nu_i + 1) I_5^{[d+]} = \frac{1}{()_5} \left[\binom{0}{0}_5 - \sum_{k=1}^{5} \binom{0}{k}_5 \mathbf{k}^- \right] I_5, \tag{18}$$

where the operators $\mathbf{i}^\pm, \mathbf{j}^\pm, \mathbf{k}^\pm$ act by shifting the indices ν_i, ν_j, ν_k by ± 1.

Dimensional shifts and recurrence relations for pentagons (III)

Represent a pentagon tensor of rank R:
After repeated use of the recurrence relations, all the higher dimensional scalar integrals disappear

- A representation by the simple scalar functions in d dimensions is achieved:
  self-energies B_0
  vertices C_0
  boxes D_0
- For the tensor rank R one gets
  inverse powers of Gram determinants: $\left(\frac{1}{()_5}\right)^R$

The algebraic derivations have to be re-organized in order to cancel in a controlled way these inverse powers of Gram determinants

The result of simplifying manipulations [indicated in the backup slides (mark 8)] ...

...and collecting all contributions, our final result for e.g. the tensor of rank $R = 3$ can be written as follows:

$$I_5^{\mu\,\nu\,\lambda} = \sum_{i,j,k=1}^{4} q_i^{\mu}\, q_j^{\nu}\, q_k^{\lambda} E_{ijk} + \sum_{k=1}^{4} g^{[\mu\nu} q_k^{\lambda]} E_{00k}, \tag{19}$$

with:

$$E_{00j} = \sum_{s=1}^{5} \frac{1}{\binom{0}{0}_5} \left[\frac{1}{2} \binom{0s}{0j}_5 I_4^{[d+],s} - \frac{d-1}{3} \binom{s}{j}_5 I_4^{[d+]^2,s} \right], \tag{20}$$

$$E_{ijk} = -\sum_{s=1}^{5} \frac{1}{\binom{0}{0}_5} \left\{ \left[\binom{0j}{sk}_5 I_{4,i}^{[d+]^2,s} + (i \leftrightarrow j) \right] + \binom{0s}{0k}_5 \nu_{ij} I_{4,ij}^{[d+]^2,s} \right\}. \tag{21}$$

✓ no scalar 5-point integrals in higher dimensions
✓ no inverse Gram det. $()_5$

We have yet:

† scalar 4-point integrals in higher dimensions: $I_{4,ij}^{[d+]^2,s}$ etc.

† inverse Gram det. $\binom{0}{0}_5 \equiv ()_4$

Isolation of inverse sub-Gram det^s $()_4$ I

We have now two kinds of objects in higher dim^s to be evaluated:

$$I_4^s,\ I_4^{[d+],s},\ I_4^{[d+]^2,s} \qquad \text{boxes} \qquad (22)$$

$$I_{4,i}^{[d+],s},\ I_{4,i}^{[d+]^2,s} \quad I_{4,ij}^{[d+]^2,s} \qquad \text{boxes with higher indices} \qquad (23)$$

Application of dimension-shifting recurrence relations produces powers of $1/()_4$.
They will be the unwanted and unavoidable sub-Gram-determinants $()_4$.
Next – and last – two steps:

- Reduce the $I_{4,i}^{[d+],s}$, $I_{4,i}^{[d+]^2,s}$, $I_{4,ij}^{[d+]^2,s}$ etc. to non-indexed scalars
- Then look at the non-indexed scalars

Reduce $I_{4,ij\cdots}^{[d+]^l,s}$ to $I_4^{[d+]^L,s}$ plus simpler objects I

By nontrivial manipulations we get e.g.:

$$I_{4,i}^{[d+],s} = \frac{1}{\binom{0s}{0s}_5}\left[-\binom{0s}{is}_5 (d-3) I_4^{[d+],s} + \sum_{t=1}^{5} \binom{0st}{0si}_5 I_3^{st}\right] \quad (24)$$

$$\begin{aligned} \nu_{ij} I_{4,ij}^{[d+]^2} &= \frac{\binom{0}{i}_4}{\binom{0}{0}_4}\frac{\binom{0}{j}_4}{\binom{0}{0}_4}(d-2)(d-1) I_4^{[d+]^2} + \frac{\binom{0i}{0j}_4}{\binom{0}{0}_4} I_4^{[d+]} \\ &- \frac{\binom{0}{j}_4}{\binom{0}{0}_4}\frac{d-2}{\binom{0}{0}_4}\sum_{t=1}^{4}\binom{0t}{0i}_4 I_3^{[d+],t} + \frac{1}{\binom{0}{0}_4}\sum_{t=1}^{4}\binom{0t}{0j}_4 I_{3,i}^{[d+],t} \quad (25) \end{aligned}$$

These equations are free of inverse Gram determinants $()_4$.
But they contain yet the generic 4-point and (partly indexed) 3-point functions in higher dimensions, $I_4^{[d+],s}$, $I_3^{[d+],t}$, etc.

Last step: evaluate the $I_4^{[d+],s}$, $I_3^{[d+],t}$, etc. I

Several strategies are now possible:

- Just evaluate them analytically in $d + 2l - 2\epsilon$ dimensions – if you may do that
- Just evaluate them numerically in $d + 2l - 2\epsilon$ dimensions
- Reduce them further by recurrences – buy the towers of $1/()_4$ $\rightarrow$ apply (18)
- Make a small Gram determinant expansion $\rightarrow$ apply (18) another way round

Last two items are done here.

Reduction of scalars I_4^D to the generic dimension $\to I_4^d = D_0, I_3^d = C_0$ I

Non-small 4-point Gram determinants:
Direct, iterative use of (18) yields e.g.:

$$I_4^{[d+]^l} = \left[\frac{\binom{0}{0}_4}{()_4} I_4^{[d+]^{l-1}} - \sum_{t=1}^{4} \frac{\binom{t}{0}_4}{()_4} I_3^{[d+]^{l-1},t} \right] \frac{1}{d+2l-5} \qquad (26)$$

$$I_3^{[d+]^l,t} = \left[\frac{\binom{0t}{0t}_4}{\binom{t}{t}_4} I_3^{[d+]^{l-1},t} - \sum_{u=1,u\neq t}^{4} \frac{\binom{ut}{0t}_4}{\binom{t}{t}_4} I_2^{[d+]^{l-1},tu} \right] \frac{1}{d+2l-4} \qquad (27)$$

And we are done.
This works fine if $()_4$ is not small.

Make a small Gram expansion I

Again use (18):

$$()_4 (d - \sum_{i=1}^{4} \nu_i + 1) I_4^{[d+]} = \left[\binom{0}{0}_4 I_4 - \sum_{k=1}^{4} \binom{0}{k}_4 I_3^k \right]$$

If $()_4 = 0$, then it follows ($n = 4$):

$$I_n^D = \sum_k^n \frac{\binom{0}{k}_n}{\binom{0}{0}_n} I_{n-1}^{D,k} \qquad (28)$$

If $()_4 << 1$, re-write (18), as follows:

$$I_n^D = \sum_k^n \frac{\binom{0}{k}_n}{\binom{0}{0}_n} I_{n-1}^{D,k} - \frac{()_n}{\binom{0}{0}_n} [(D+1) - \sum_i^n \nu_i] I_n^{D+2}. \qquad (29)$$

Effectively we may evaluate I_n^D in terms of simpler functions $I_{n-1}^{D,k}$ with a small correction depending on I_n^{D+2}.

We may go a step further, and insert into (29) for I_n^{D+2} the rhs. of (28), taken now at $D' = D+2$:

$$I_n^D = \sum_k^n \frac{\binom{0}{k}_n}{\binom{0}{0}_n} I_{n-1}^{D,k} - \frac{()_n}{\binom{0}{0}_n}[(D+1) - \sum_i^n \nu_i] \times \left[\sum_k^n \frac{\binom{0}{k}_n}{\binom{0}{0}_n} I_{n-1}^{D+2,k} - \frac{()_n}{\binom{0}{0}_n}[(D+3) - \sum_i^n \nu_i] I_n^{D+4} \right].$$

The terms proportional to $[()_n/\binom{0}{0}_n]^a$, $a = 0, 1$ may be evaluated at the correct kinematics. They depend on three-point functions, and their reduction by normal recurrences will not introduce the unwanted powers of $1/()_4$. The last term, suppressed by the factor $[()_n/\binom{0}{0}_n]^2$, depends on I_n^{D+4}. It may either be taken approximately at $()_n = 0$, where it can also be represented by 3-point functions (and their reductions), or it may be evaluated more correctly by another iteration based on (28).

And so on and so on

In the numerical example – next section – we worked out up to 10 stable iterations.

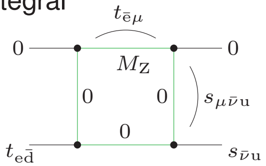
A quite similar attempt to perform such a series of approximations was undertaken in [15] (see equation (5) there), where a specific example, forward light-by-light scattering through a massless fermion loop, was studied. The approach was then not further followed.

W. Giele, E. W. N. Glover, and G. Zanderighi,
in: Proceedings of Loops ans Legs 2004:
Numerical evaluation of one-loop diagrams near exceptional momentum configurations,

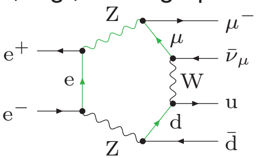
An example from A. Denner [11]: 4-point tensor of rank 3 D_{111}

Few figures copied from: A.Denner, plenary talk DESY Theory Workshop 2009, p.69 (backup transparency)

box integral



appears, e.g., in subgraph of diagram

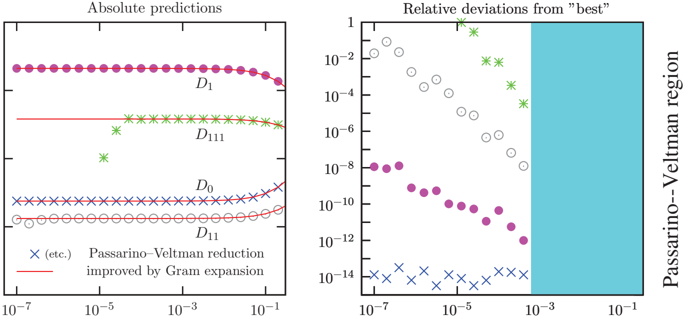


Gram det.: $\Delta^{(N)} \to 0$ if $t_{e\bar{d}} \to t_{\rm crit} \equiv \dfrac{s_{\mu\bar{\nu}u}(s_{\mu\bar{\nu}u} - s_{\bar{\nu}u} + t_{\bar{e}\mu})}{s_{\mu\bar{\nu}u} - s_{\bar{\nu}u}}$

The figure demonstrates the effects of careful treatment of vanishing Gram determinant $()_4$.

Gram det.: $\Delta^{(N)} \to 0$ if $t_{e\bar{d}} \to t_{crit} \equiv \frac{s_{\mu\bar{\nu}u}(s_{\mu\bar{\nu}u} - s_{\bar{\nu}u} + t_{\bar{e}\mu})}{s_{\mu\bar{\nu}u} - s_{\bar{\nu}u}}$

numerical comparison: maximal tensor rank $= 6$ (similar to $ee \to 4f$ application)



$x \equiv \frac{t_{e\bar{d}}}{t_{crit}} - 1$

$$\begin{aligned} s_{\mu\bar{\nu}u} &= +2\times10^4\ \mathrm{GeV}^2 \\ s_{\bar{\nu}u} &= +1\times10^4\ \mathrm{GeV}^2 \\ t_{\bar{e}\mu} &= -4\times10^4\ \mathrm{GeV}^2 \\ t_{crit} &= -6\times10^4\ \mathrm{GeV}^2 \end{aligned}$$

PV reduction breaks down,
but Gram exp. stable
for $\Delta^{(N)} \to 0$!

Following Davydychev, [5], one gets

$$I_4^{\mu\nu\lambda} \quad = \quad \int^d \frac{k^\mu k^\nu k^\lambda}{\prod_{r=1}^n c_r} = -\sum_{i,j,k=1}^{n} q_i^\mu q_j^\nu q_k^\lambda \nu_{ijk} I_{n,ijk}^{[d+]^3} + \frac{1}{2}\sum_{i=1}^{n} g^{[\mu\nu} q_i^{\lambda]} I_{n,i}^{[d+]^2} \quad (30)$$

We identify the tensor coefficient D_{111} a la LoopTools:

$$D_{111} \sim \nu_{ijk} I_{4,ijk}^{[d+]^3} \quad \text{for} \quad ijk = 222$$

Rank $R = 4$ tensor D_{1111} – Numerics with dimensional recurrences

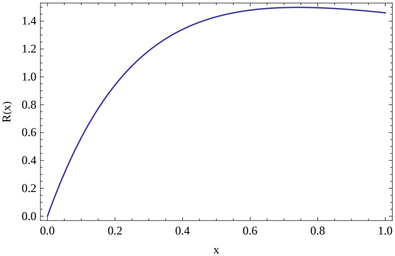
From (29) we see that a “small Gram determinant” expansion will be useful when the following dimensionless parameter becomes small:

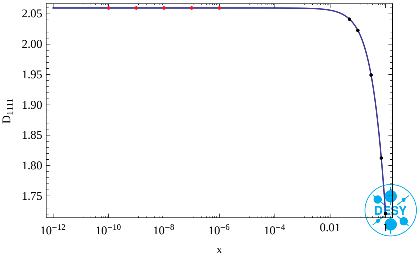
$$R = \frac{()_4}{\binom{0}{0}_4} \times s, \tag{31}$$

where s is a typical scale of the process, e.g. we will choose $s = s_{\mu\bar{\nu}u}$.
Following [11], we further choose:

$$\begin{aligned} s_{\mu\bar{\nu}u} &= 2 \times 10^4 \mathrm{GeV}^2, \\ s_{\bar{\nu}u} &= 1 \times 10^4 \mathrm{GeV}^2, \\ t_{\bar{e}\mu} &= -4 \times 10^4 \mathrm{GeV}^2, \end{aligned}$$

and get $t_{ed,\mathrm{crit}} = -6 \times 10^4 \mathrm{GeV}^2$. For x=1, the Gram determinant becomes $()_4 = 4.8 \times 10^{13}$ GeV^3.
The small expansion parameter $R(x)$ and D_{1111} are shown in figure 2.





Rank $R = 4$ tensor D_{1111} – Numerics with dimensional recurrences

x		$\Re e\, D_{1111}$	$\Im m\, D_{1111}$	$\Re e$
0.	[exp0]	2.059692897296995 E-10	1.555949101177984 E-10	
10^{-8}	[exp2]	2.0596928934853468 E-10	1.55594909187293 E-10	
10^{-4}	[exp5]	2.05965609495210 E-10	1.555856053429301 E-10	
0.001	[exp6]	2.0593248437953651 E-10	1.555019124326089 E-10	
0.001	[pade]	2.0593248436598399 E-10	1.5550191243261055 E-10	
0.001	[direct]	2.0229229523996894 E-10	1.5497478546690215 E-10	
0.005	[exp6]	2.0578605480053023 E-10	1.5513103102416075 E-10	
0.005	[pade]	2.0578519894658186 E-10	1.5513103100323308 E-10	
0.005	[direct]	2.0577889411443721 E-10	1.551357944527207 E-10	
0.01	[exp6]	2.0570329814337165 E-10	1.5466991067608538 E-10	
0.01	[pade]	2.0560095165549248 E-10	1.5466994087841823 E-10	
0.01	[fit5]	2.0560093196591156 E-10		
0.01	[direct]	2.056000106408516 E-10	1.546706521399316 E-10	
0.01	[LoopT]	2.0560023928083998 E-10	1.5467077121032603 E-10	
0.05	[exp6]	4.838229630519484 E-09	1.5107742912166673 E-10	
0.05	[pade]	2.015180611305954 E-10	1.5059164320937378 E-10	
0.05	[direct]	2.0412272638658917 E-10	1.5107742290135455 E-10	
0.05	[LoopT]	2.041227266007564 E-10	1.5107742332021534 E-10	

Rank $R = 3$ D_{111} – Numerics with dimensional recurrences

x	$\Re e\ D_{111}$	$\Im m\ D_{111}$	
0 [exp0]	–3.154072504525619 E-10	–3.318377926336023 E-10	
10^{-8} [exp1]	–3.1540725005731514 E-10	–3.3183779070041916 E-10	
10^{-4} [exp4]	–3.1540328219426004 E-10	–3.318184618382335 E-10	
0.001 [exp6]	–3.1536754542867605 E-10	–3.316445871504251 E-10	
0.001 [pade]	–3.1536754542867605 E-10	–3.3164458715042346 E-10	
0.001	–3.1537209279927465 E-10	–3.3164524564412596 E-10	–3.
0.005 [exp6]	–3.1520822486710397 E-10	–3.3087403586191434 E-10	
0.005 [pade]	–3.1520823041125224 E-10	–3.308740358668981 E-10	
0.005	–3.152082697913492 E-10	–3.308740061095388 E-10	–3.
0.01 [exp6]	–3.150066 652840638 E-10	–3.2991592611039606 E-10	
0.01 [pade]	–3.1500797 783006643 E-10	–3.2991588807525176 E-10	
0.01	–3.1500800312554073 E-10	–3.299159168481735 E-10	–3.
0.05 [exp6]	–1.3427847021090757 E-11	–3.2244858072157833 E-10	
0.05 [pade]	–3.134325165703912 E-10	–3.22580791798769 E-10	
0.05	–3.1336567508368213 E-10	–3.224485811098255 E-10	–3.
0.1	–3.1122675069886563 E-10	–3.1358233197652523 E-10	–3.
1.	–2.701937913717525 E-10	–2.1025197382076437 E-10	–2.

Rank $R=2$ D_{11} etc. – Numerics based on the dimensional recurrences

x	$\Re e\, D_{11}$	$\Im m\, D_{11}$	
0 [exp0]	4.696899199595157 E-10	7.524590274244601 E-10	
10^{-8} [exp0]	4.696899199352371 E-10	7.524590232156507 E-10	
10^{-4} [exp2]	4.696896808117155 E-10	7.524169417021979 E-10	
0.001 [exp6]	4.696874404573178 E-10	7.5203838941417 5 E-10	
0.001 [pade]	4.696874404573193 E-10	7.52038389414175 E-10	
0.001	4.696874347766776 E-10	7.520383885915733 E-10	4.696874
0.005 [exp6]	4.696755776545425 E-10	7.503606949514908 E-10	
0.005 [pade]	4.696755776182078 E-10	7.503606949514908 E-10	
0.005	4.696755773722263 E-10	7.503606951373359 E-10	4.696755
0.01 [exp6]	4.696564420190031 E-10	7.482744433740568 E-10	
0.01 [pade]	4.696564330848334 E-10	7.482744525718862 E-10	
0.01	4.69656424511161 E-10	7.482744434897886 E-10	4.696564
0.05 [exp6]	4.696564420190031 E-10	7.482744433740568 E-10	
0.05 [pade]	4.696564330848334 E-10	7.482744525718862 E-10	
0.05	4.693398532541272 E-10	7.320070719323306 E-10	4.693398
0.1	4.685784140044507 E-10	7.126742971329895 E-10	4.685784
1.	4.275782386841888 E-10	4.854786682396297 E-10	4.275782

Summary

- Recursive treatment of hexagon and pentagon tensor integrals of rank R in terms of pentagons and boxes of rank $R-1$
- Systematic derivation of expressions which are explicitely free of inverse Gram determinants $()_5$ until pentagons of rank $R=5$
- Proper isolation of inverse Gram determinants of subdiagrams of the type $\binom{s}{s}_n$, which cannot be completely avoided
- Some numerics, so far in Mathematica, and a numerical C++ package (together with V. yundin) under way

References I

- G. Passarino and M. Veltman, *One loop corrections for e^+e^- annihilation into $\mu^+\mu^-$ in the Weinberg model*, *Nucl. Phys.* **B160** (1979) 151.
- G. 't Hooft and M. Veltman, *Scalar one loop integrals*, *Nucl. Phys.* **B153** (1979) 365–401.
- T. Hahn and M. Perez-Victoria, *Automatized one-loop calculations in four and d dimensions*, *Comput. Phys. Commun.* **118** (1999) 153, [hep-ph/9807565].
- G. J. van Oldenborgh, *FF: A Package to evaluate one loop Feynman diagrams*, *Comput. Phys. Commun.* **66** (1991) 1–15.
- A. I. Davydychev, *A simple formula for reducing Feynman diagrams to scalar integrals*, *Phys. Lett.* **B263** (1991) 107–111.
- O. Tarasov, *Connection between Feynman integrals having different values of the space-time dimension*, *Phys. Rev.* **D54** (1996) 6479–6490, [hep-th/9606018].
- J. Fleischer, F. Jegerlehner, and O. Tarasov, *Algebraic reduction of one-loop Feynman graph amplitudes*, *Nucl. Phys.* **B566** (2000) 423–440, [hep-ph/9907327].
- T. Diakonidis, J. Fleischer, J. Gluza, K. Kajda, T. Riemann, and J. Tausk, *A complete reduction of one-loop tensor 5- and 6-point integrals*, *Phys. Rev.* **D80** (2009) 036003, [arXiv:0812.2134].
- T. Diakonidis, J. Fleischer, T. Riemann, and J. B. Tausk, *A recursive reduction of tensor Feynman integrals*, *Phys. Lett.* **B683** (2010) 69–74, [arXiv:0907.2115].

References II

- J. Fleischer and T. Riemann, *Some variations of the reduction of one-loop Feynman tensor integrals*, *PoS* **RADCOR2010** (2010) 074, [arXiv:1006.0679].
- A. Denner, *Techniques and concepts for higher order calculations*, Introductory Lecture at DESY Theory Workshop on Collider Phenomenology, Hamburg, 29 Sep - 2 Oct 2009.
- T. Hahn, *LoopTools 2.5 User's Guide*, LT25Guide.pdf.
- T. Binoth, J. Guillet, G. Heinrich, E. Pilon, and C. Schubert, *An algebraic / numerical formalism for one-loop multi-leg amplitudes*, *JHEP* **10** (2005) 015, [hep-ph/0504267].
- A. Denner and S. Dittmaier, *Reduction schemes for one-loop tensor integrals*, *Nucl. Phys.* **B734** (2006) 62–115, [hep-ph/0509141].
- W. Giele, E. W. N. Glover, and G. Zanderighi, *Numerical evaluation of one-loop diagrams near exceptional momentum configurations*, *Nucl. Phys. Proc. Suppl.* **135** (2004) 275–279, [hep-ph/0407016].
- T. Binoth, J. P. Guillet, G. Heinrich, E. Pilon, and T. Reiter, *Golem95: a numerical program to calculate one-loop tensor integrals with up to six external legs*, *Comput. Phys. Commun.* **180** (2009) 2317–2330, [0810.0992].
- T. Riemann, G. Mann, and D. Ebert.
  Nonconservation of lepton number in Z decay, in: F. Kaschluhn (ed.), Proc. *XVth Int. Symp. Ahrenshoop on Special Topics In Gauge Field Theories*, Nov 5-12, 1981, Ahrenshoop, GDR, AdW, Zeuthen (1981) PHE 81-07, pp. 88-91.

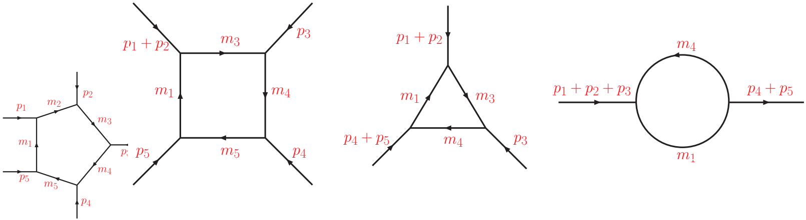
References III

- G. Mann and T. Riemann, *Effective flavor changing weak neutral current in the standard theory and Z boson decay*, *Annalen Phys.* **40** (1984) 334.
- J. Fleischer and F. Jegerlehner, *Radiative Corrections to Higgs Decays in the Extended Weinberg-Salam Model*, *Phys. Rev.* **D23** (1981) 2001–2026.
- T. Diakonidis, J. Fleischer, J. Gluza, K. Kajda, T. Riemann, and J. Tausk, *On the tensor reduction of one-loop pentagons and hexagons*, *Nucl. Phys. Proc. Suppl.* **183** (2008) 109–115, [0807.2984].
- R. K. Ellis and G. Zanderighi, *Scalar one-loop integrals for QCD*, *JHEP* **02** (2008) 002, [arXiv:0712.1851].
- J. Campbell, E. W. N. Glover, and D. Miller, *One-loop tensor integrals in dimensional regularisation*, *Nucl. Phys.* **B498** (1997) 397–442, [hep-ph/9612413].
- A. Denner and S. Dittmaier, *Reduction of one-loop tensor 5-point integrals*, *Nucl. Phys.* **B658** (2003) 175–202, [hep-ph/0212259].
- T. Binoth, J. P. Guillet, and G. Heinrich, *Reduction formalism for dimensionally regulated one-loop N-point integrals*, *Nucl. Phys.* **B572** (2000) 361–386, [hep-ph/9911342].
- Z. Bern, L. J. Dixon, and D. A. Kosower, *Dimensionally regulated pentagon integrals*, *Nucl. Phys.* **B412** (1994) 751–816, [hep-ph/9306240].
- G. Ossola, C. Papadopoulos, and R. Pittau, *Reducing full one-loop amplitudes to scalar integrals at the integrand level*, *Nucl. Phys.* **B763** (2007) 147–169, [hep-ph/0609007].

Numbers (I) – Pentagons

Randomly chosen phase space point with massive and massless internal particles

p_1	5.00000000000 E+00	0.0000000000 E+00	0.00000000000 E+00	4.00000000000 E+00
p_2	5.00000000000 E+00	0.0000000000 E+00	0.00000000000 E+00	-4.00000000000 E+00
p_3	– 0.30770034895 E+01	0.5359484673 E+00	–0.37447035150 E+00	– 0.20120057390 E+00
p_4	– 0.34048537280 E+01	0.2184763540 E-01	– 0.10479394969 E+01	0.12224460727 E+01
p_5	– 0.35181427825 E+01	– 0.5577961027 E+00	0.14224098484 E+01	– 0.10212454988 E+01
$m_1 = 0.0, \quad m_2 = 2.0, \quad m_3 = 3.0, \quad m_4 = 4.0, \quad m_5 = 5.0$				



Selected pentagon components

Shown are the constant terms of the tensor components

	Pentagon.F
E^2	(2.80450709388539E-05 , -1.08461817406464E-05)
E^{12}	(-5.41333978667301E-06 , 6.26985967678899E-06)
E^{232}	(-1.20374858970726E-04 , 4.07974751672555E-04)
E^{0321}	(-9.11194535703727E-06 , 4.39187998675819E-05)
E^{01230}	(4.37928367160152E-05 , -2.18183151665913E-04)

Box.F	*LoopTools*
(6.81403420828588E-03 , -5.74298462683219E-03)	(6.8140342082847463E-03 , -5.7429846268324187E-03)
(2.40138809967981E-03 , 1.11591328775015E-02)	(2.4013880996803092E-03,1.1159132877500448E-02)
(-1.69702786278243E-03 , -2.83731121595478E-03)	(-1.6970278627700630E-03,-2.8373112159962330E-03)
(-1.92190388316994E-04 , -4.04730302413490E-04)	(-1.9219038693301300E-04,-4.0473030187772325E-04)

	Triangle.F	*LoopTools*
C^2	(2.44757827793318E-04 , -7.50688449850356E-03)	(2.4475782779342707E-04,-7.5068844985030472E-03)
C^{01}	(-1.28259813172255E-02 , -6.73809718907549E-02)	(-1.2825981317215014E-02,-6.7380971890795340E-02
C^{133}	(-7.00360822297110E-02 , 7.24628606014397E-02)	(-7.0036082229746830E-02,7.2462860601566081E-02)

	Bubble.F	*LoopTools*
B^3	(-0.141525070262337E+00 , 0.138870631815383E+00)	(-0.1415250702623366,0.1388706318153829)
B^{12}	(0.102490343329085E+00 , -6.12154531068256E-02)	(0.1024903433290848,-6.1215453106825706E-02)

here some text 1.

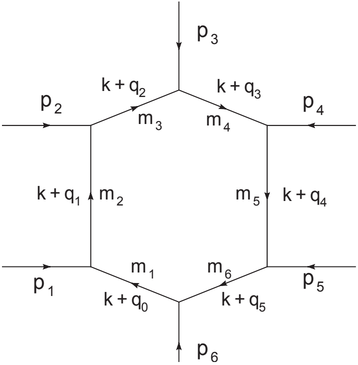


Figure: Momenta flow for the massive six-point topology.

here some text 2.

Numbers (II) – Hexagons

p_1	0.21774554 E+03	0.0	0.0	0.21774554 E+03
p_2	0.21774554 E+03	0.0	0.0	– 0.21774554 E+03
p_3	– 0.20369415 E+03	– 0.47579512 E+02	0.42126823 E+02	0.84097181 E+02
p_4	– 0.20907237 E+03	0.55215961 E+02	– 0.46692034 E+02	– 0.90010087 E+02
p_5	– 0.68463308 E+01	0.53063195 E+01	0.29698267 E+01	– 0.31456871 E+01
p_6	– 0.15878244 E+02	– 0.12942769 E+02	0.15953850 E+01	0.90585932 E+01
$m_1 = 110.0, \; m_2 = 120.0, \; m_3 = 130.0, \; m_4 = 140.0, \; m_5 = 150.0, \; m_6 = 160.0$				

		F_0
		– 0.223393 E–18 – i 0.396728 E–19
μ		F^{μ}
0		0.192487 E–17 + i 0.972635 E–17
1		– 0.363320 E–17 – i 0.11940 E–17
2		0.365514 E–17 + i 0.106928 E–17
3		0.239793 E–16 + i 0.341928 E–17
μ	ν	$F^{\mu\nu}$
0	0	0.599459 E–14 – i 0.114601 E–14
0	1	0.323869 E–15 + i 0.423754 E–15
0	2	– 0.294252 E–15 – i 0.375481 E–15
0	3	– 0.255450 E–14 – i 0.195640 E–14
1	1	– 0.164562 E–14 – i 0.993796 E–16
1	2	0.920944 E–16 + i 0.706487 E–17
1	3	0.347694 E–15 – i 0.127190 E–16
2	2	– 0.163339 E–14 – i 0.994148 E–16
2	3	– 0.341773 E–15 + i 0.818678 E–17
3	3	– 0.413909 E–14 + i 0.670676 E–15

μ	ν	λ	$F^{\mu\nu\lambda}$
0	0	0	– 0.227754 E–11 – i 0.267244 E–12
0	0	1	0.140271 E–13 – i 0.119448 E–12
0	0	2	– 0.201270 E–13 + i 0.101968 E–12
0	0	3	0.102976 E–12 + i 0.624467 E–12
0	1	1	0.183904 E–12 + i 0.142429 E–12
0	1	2	– 0.131028 E–13 – i 0.610343 E–14
0	1	3	– 0.543316 E–13 – i 0.158809 E–13
0	2	2	0.181352 E–12 + i 0.141686 E–12
0	2	3	0.506408 E–13 + i 0.163568 E–13
0	3	3	0.600542 E–12 + i 0.130733 E–12
1	1	1	– 0.563539 E–13 + i 0.178403 E–13
1	1	2	0.210641 E–13 – i 0.584990 E–14
1	1	3	0.120482 E–12 – i 0.574688 E–13
1	2	2	– 0.201182 E–13 + i 0.620591 E–14
1	2	3	– 0.686164 E–14 + i 0.205457 E–14
1	3	3	– 0.447329 E–13 + i 0.193180 E–13
2	2	2	0.582201 E–13 – i 0.163889 E–13
2	2	3	0.119659 E–12 – i 0.570084 E–13
2	3	3	0.457464 E–13 – i 0.181141 E–13
3	3	3	0.557081 E–12 – i 0.374359 E–12

Table: Tensor components for a massive rank $R = 3$ six-point function

μ	ν	λ	ρ	$F^{\mu\nu\lambda\rho}$
0	0	0	0	0.666615 E–09 + i 0.247562 E–09
0	0	0	1	– 0.200049 E–10 + i 0.294036 E–10
0	0	0	2	0.200975 E–10 – i 0.237333 E–10
0	0	0	3	0.645477 E–10 – i 0.162236 E–09
0	0	1	1	– 0.116956 E–10 – i 0.516760 E–10
0	0	1	2	0.160357 E–11 + i 0.222284 E–11
0	0	1	3	0.792692 E–11 + i 0.729502 E–11
0	0	2	2	– 0.111838 E–10 – i 0.513133 E–10
0	0	2	3	– 0.681086 E–11 – i 0.708933 E–11
0	0	3	3	– 0.804454 E–10 – i 0.801909 E–10
0	1	1	1	0.100498 E–10 – i 0.151735 E–13
0	1	1	2	– 0.348984 E–11 – i 0.195436 E–12
0	1	1	3	– 0.211111 E–10 + i 0.295212 E–11
0	1	2	2	0.357455 E–11 + i 0.662809 E–14
0	1	2	3	0.121595 E–11 – i 0.807388 E–13
0	1	3	3	0.825803 E–11 – i 0.142086 E–11
0	2	2	2	– 0.958961 E–11 – i 0.585948 E–12
0	2	2	3	– 0.209232 E–10 + i 0.289031 E–11
0	2	3	3	– 0.802359 E–11 + i 0.994701 E–12
0	3	3	3	– 0.102576 E–09 + i 0.378476 E–10
1	1	1	1	– 0.246426 E–10 + i 0.276326 E–10
1	1	1	2	0.915670 E–12 – i 0.660629 E–12
1	1	1	3	0.303529 E–11 – i 0.287480 E–11
1	1	2	2	– 0.822697 E–11 + i 0.919635 E–11
1	1	2	3	– 0.116294 E–11 + i 0.100024 E–11
1	1	3	3	– 0.146918 E–10 + i 0.183799 E–10
1	2	2	2	0.908296 E–12 – i 0.654735 E–12
1	2	2	3	0.109510 E–11 – i 0.100875 E–11
1	2	3	3	0.717342 E–12 – i 0.557293 E–12
1	3	3	3	0.450661 E–11 – i 0.485065 E–11
2	2	2	2	– 0.245154 E–10 + i 0.274313 E–10
2	2	2	3	– 0.318500 E–11 + i 0.279750 E–11
2	2	3	3	– 0.146317 E–10 + i 0.182912 E–10
[illegible]	[illegible]	[illegible]	[illegible]	0.477225 E–11 + i 0.477268 E–11

p_1	0.5	0.0	0.0	0.5
p_2	0.5	0.0	0.0	− 0.5
p_3	− 0.19178191	− 0.12741180	− 0.08262477	− 0.11713105
p_4	− 0.33662712	0.06648281	0.31893785	0.08471424
p_5	− 0.21604814	0.20363139	− 0.04415762	− 0.05710657
$p_6 = -(p_1 + p_2 + p_3 + p_4 + p_5)$				

Table: Phase space point of massless six-point functions taken from [Binoth:2008 [16]] . Golem95: Binoth, Guillet, Heinrich, Pilon, Reiter [arXiv:hep-ph/0810.0992]

Shown are only the constant terms of the tensor components.

	Hexagon.F	*Golem95*
F^{03121}	(0.158428986740235E+00 , 0.416706979843194E-01)	(0.158428980552600E+00 , 0.416706995132716E-01)
F^{11020}	(-0.143913859903552E+01 , -0.164647048275408E+00)	(-0.143913852754709E+01 , -0.164647075385477E+00)
F^{20200}	(0.242928799509288E+02 , 0.555041844207877E+02)	(0.242928775936564E+02 , 0.555041824180155E+02)
F^{22130}	(0.225563941055782E+00 , 0.231928571404353E+00)	(0.225563949300093E+00 , 0.231928509918651E+00)
F^{33333}	(0.244568134868438E+00 , 0.740146041525474E+00)	(0.244568138432017E+00 , 0.740146095196997E+00)

Algebraic simplifications, 1st step

With the identity

$$\binom{0}{0}_5 \binom{s}{i}_5 = \binom{0s}{0i}_5 ()_5 + \binom{0}{i}_5 \binom{s}{0}_5 \qquad (32)$$

we eliminate the inverse Gram determinant from all terms with exclusion of Q_0^μ:

$$I_5^{\mu_1\cdots\mu_{R-1}\mu} = \left[I_5^{\mu_1\cdots\mu_{R-1}} - \sum_{s=1}^{5} \frac{\binom{s}{0}_5}{\binom{0}{0}_5} I_4^{\mu_1\cdots\mu_{R-1},s} \right] Q_0^\mu - \sum_{s=1}^{5} I_4^{\mu_1\cdots\mu_{R-1},s} \overline{Q}_s^\mu \qquad (33)$$

The auxiliary vectors $\overline{Q}_s^\mu$ were introduced already for $n = 6$:

$$Q_0^\mu = \sum_{i=1}^{5} q_i^\mu \frac{\binom{0}{i}_5}{()_5} \quad \text{while} \quad \overline{Q}_s^\mu = \sum_{i=1}^{5} q_i^\mu \frac{\binom{0s}{0i}_5}{\binom{0}{0}_5} \qquad (34)$$

Algebraic simplifications, 2nd step I

Have to show for the product $T^{\mu_1\cdots\mu_{R-1}} \times Q_0^{\mu}$ that the Gram determinant cancels.
This came out to be a complicated task.

$$T^{\mu_1\cdots\mu_{R-1}} = \left[\binom{0}{0}_5 I_5^{\mu_1\cdots\mu_{R-1}} - \sum_{s=1}^{5} \binom{s}{0}_5 I_4^{\mu_1\cdots\mu_{R-1},s} \right] \qquad (35)$$

Example: For $R = 3$ pentagons need rank 2 tensor:

$$T^{\mu\nu} = \left[\binom{0}{0}_5 I_5^{\mu\nu} - \sum_{s=1}^{5} \binom{s}{0}_5 I_4^{\mu\nu,s} \right] \qquad (36)$$

Algebraic simplifications, 2nd step I

Example $R = 3$: the building blocks are here $I_5^{\mu\nu}$ and $I_4^{\mu\nu}$:

$$I_5^{\mu\nu} = \sum_{i,j=1}^{5} q_i^\mu q_j^\nu \left[(1+\delta_{ij}) I_{5,ij}^{[d+]^2}\right] + g^{\mu\nu}\left[-\frac{1}{2} I_5^{[d+]}\right]$$

$$\stackrel{\text{work!!!}}{\Rightarrow} \sum_{i,j=1}^{4} q_i^\mu q_j^\nu \frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5}\left[\binom{0i}{sj}_5 I_4^{[d+],s} + \binom{0s}{0j}_5 I_{4,i}^{[d+],s}\right]$$

$$+ g^{\mu\nu}\left[-\frac{1}{2}\frac{1}{\binom{0}{0}_5}\sum_{s=1}^{5}\binom{s}{0}_5 I_4^{[d+],s}\right] \qquad (37)$$

See: The $I_5^{\mu\nu}$ had already been made free of $1/()_5$.

Davydychev's higher dimensional integrals

The second term with $I_4^{\mu\nu}$ is a typical example of [Davydychev:1991 [5]] : tensor $\Rightarrow$ scalars in $d+2l$

$$I_4^{\mu\nu} = \sum_{i,j=1}^{4} q_i^\mu q_j^\nu \left[(1+\delta_{ij}) I_{4,ij}^{[d+]^2} \right] + g^{\mu\nu} \left[-\frac{1}{2} I_4^{[d+]} \right] \quad (38)$$

Further, at this point, we have to reduce the scalar integrals $I_{4,ij}^{[d+]^2}$ etc. to generic dimension d with Tarasov's recurrence relations, see next slide.
The $I_4^{\mu\nu}$ is naturally free of $1/()_5$.

Reduction of tensor integrals ⇒ express them by a (very) small set of scalar integrals

Presently needed for massive processes:

$n \leq 6$ and rank $R \leq n$

For box diagrams and simpler ones:
Use of the 'conventional' Passarino-Veltman reduction
[Passarino:1978jh [1]]

Examples:

- LO (Lowest order) of e.g. $Z \to e + \mu$ is one-loop [Riemann:1981 [17], Mann:1983 [18]]
- NLO: one-loop corrections to e.g. $H \to \tau^+\tau^-, WW, ZZ$ [Fleischer:1980 [19]]
- NNLO: e.g. radiative loop corrections $e^+e^- \to e^+e^-\gamma$ (here with 5-point functions)

Some opensource packages

- package FF [vanOldenborgh:1990 [4]]
- package LoopTools/FF v.2

  [Hahn:1998,2006 [3]] – covers also 5-point functions, rank $R \leq 4$
  $1/\epsilon^2$ not covered, and we observed sometimes problems in certain configurations with light-like external particles
- package Golem95 [Binoth:2008 [16]] for $n \leq 6$, but only massless propagators
- Mathematica package hexagon.m [Diakonidis:2008 [20, 8]] for $n \leq 6$, rank $R \leq 4$
- package for all $n \leq 4$ scalar integrals: QCDloop [Ellis:2007 [21]]
- see also: review A.Denner, DESY TH workshop 2009
- Our approach:
  Package hexagon.m by Kajda et al.
  Package olotic.F by Diakonidis et al.
  In preparation: C++ package fry by V. Yundin et al.

Crucial contributions [of course, list is incomplete . . .] ⇒

- [Campbell:1996 [22]]
- [Denner:2002,2005 [23, 14]]
- [Binoth:1999,2005 [24, 13]]
- [Bern:1993 [25]]
- [Ossola:2006 [26]]

In the following, I will describe recent developments in the Fleischer-Davydychev-Tarasov approach.

- [Davydychev:1991, Tarasov:1996, Fleischer:1999,Diakonidis:2008,2009 [5, 6, 7, 8, 9]]

- get tensor reduction
- kill pentagon-Gram det's
- treat sub-Gram det's

Algebraic simplifications, 2nd step

Work out the red part in

$$\binom{0}{0}_5 I_5^{\mu\nu\lambda} = \left[\binom{0}{0}_5 I_5^{\mu\nu} - \sum_{s=1}^{5} \binom{s}{0}_5 I_4^{\mu\nu,s} \right] Q_0^{\lambda} - \sum_{s=1}^{5} I_4^{\mu\nu,s} \overline{Q}_s^{0,\lambda}$$

$\Rightarrow$ work!!! **Use of identities for the determinants**

$$\binom{0}{0}_5 \binom{s}{i}_5 = \binom{0s}{0i}_5 ()_5 + \binom{0}{i}_5 \binom{s}{0}_5 \tag{39}$$

$$\binom{s}{i}_5 \frac{\binom{0}{j}_5}{()_5} = -\binom{0i}{sj}_5 + \binom{s}{0}_5 \frac{\binom{i}{j}_5}{()_5}, \qquad g^{\mu\nu} = 2 \sum_{i,j=1}^{4} \frac{\binom{i}{j}_5}{()_5} q_i^{\mu} q_j^{\nu} \tag{40}$$

$$\binom{s}{0}_5 \binom{0s}{is}_5 = \binom{s}{i}_5 \binom{0s}{0s}_5 - \binom{s}{s}_5 \binom{0s}{0i}_5 \tag{41}$$

$$\binom{s}{0}_5 \binom{ts}{js}_5 = \binom{s}{j}_5 \binom{ts}{0s}_5 - \binom{s}{s}_5 \binom{ts}{0j}_5 \tag{42}$$

Algebraic simplifications, 2nd step

⇒ Use of identities for the determinants
work!!!

$$\binom{s}{0}_5 \binom{is}{js}_5 = \binom{s}{i}_5 \binom{0s}{js}_5 + \binom{s}{s}_5 \binom{0i}{sj}_5 \tag{43}$$

$$\binom{s}{s}_5 \binom{0st}{0st}_5 = \binom{0s}{0s}_5 \binom{st}{st}_5 - \binom{ts}{0s}_5^2 \tag{44}$$

$$\left[\binom{ts}{0s}_5 \binom{ust}{jst}_5 - \binom{ts}{js}_5 \binom{ust}{0st}_5\right] \binom{s}{s}_5 = \left[\binom{ts}{0s}_5 \binom{us}{js}_5 - \binom{ts}{js}_5 \binom{us}{0s}_5\right] \binom{st}{st}_5 \tag{45}$$

$$\sum_{t=1}^{5} \binom{ts}{is}_5 = 0 \tag{46}$$

Express pentagons $I_5^{\mu}, I_5^{\mu\nu}, I_5^{\mu\nu\lambda}$ etc. by d-shifted scalar boxes I

Intermediate result with $I_4^{[d+],s}, I_{4,ij}^{[d+]^2,s}$ etc.

$$I_5^{\mu} = -\sum_{i=1}^{4}\left[\frac{1}{\binom{0}{0}_5}\sum_{s=1}^{5}\binom{0i}{0s}_5 I_4^s\right] q_i^{\mu} \tag{47}$$

$$I_5^{\mu\,\nu} = \sum_{i,j=1}^{4} q_i^{\mu}\, q_j^{\nu} E_{ij} + g^{\mu\nu} E_{00} \tag{48}$$

$$E_{ij} = \frac{1}{\binom{0}{0}_5}\sum_{s=1}^{5}\left[\binom{0i}{sj}_5 I_4^{[d+],s} + \binom{0s}{0j}_5 I_{4,i}^{[d+],s}\right] \tag{49}$$

$$E_{00} = -\frac{1}{2}\frac{1}{\binom{0}{0}_5}\sum_{s=1}^{5}\binom{s}{0}_5 I_4^{[d+],s} \tag{50}$$

Express pentagons $I_5^{\mu}, I_5^{\mu\nu}, I_5^{\mu\nu\lambda}$ etc. by d-shifted scalar boxes II

$$I_5^{\mu\nu\lambda} = \sum_{i,j,k=1}^{4} q_i^{\mu} q_j^{\nu} q_k^{\lambda} E_{ijk} + \sum_{k=1}^{4} g^{[\mu\nu} q_k^{\lambda]} E_{00k} \tag{51}$$

$$E_{ijk} = -\frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \left\{ \left[\binom{0j}{sk}_5 I_{4,i}^{[d+]^2,s} + (i \leftrightarrow j) \right] + \binom{0s}{0k}_5 \nu_{ij} I_{4,ij}^{[d+]^2,s} \right\} \tag{52}$$

$$E_{00j} = \frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \left[\frac{1}{2} \binom{0s}{0j}_5 I_4^{[d+],s} - \frac{d-1}{3} \binom{s}{j}_5 I_4^{[d+]^2,s} \right] \tag{53}$$

These presentations are evidently free of inverse Gram determinants.

OLD Numerics, 2010-01-29 : LoopTools versus our approach

Using LoopTools call and our math numerics (preliminary):

```
x      D111

 -7 : -0.007106204244698895     +0.004653980785027325 I  D0i[dd111]
      -3.15345811639208  -10    -3.318373348243635`-10 I  Z4d30,Z4d20,I4id20

 -6 : -3.2313079078584034-06    -2.8963160014947846-06 I D0i[dd111]
      -3.1479286753545824-10    -3.318332145498356 -10 I Z4d30,Z4d20,I4id20

 -5 : -5.5231182028025025-09    +3.4832284324178667-09 I D0i[dd111]
      -3.0926394107374516-10    -3.3179201270079527-10 I Z4d30,Z4d20,I4id20

 x< -4:        LoopTools dies out

 -4 : -3.1544928789869657-10    -3.33218368329059  -10 I D0i[dd111]
      -3.0798250216856066-10    -3.3447698103297804-10 I   flei

 x < -3:       loss of accuracy

 -3 : -3.153742175665908 -10    -3.31639655233478  -10 I D0i[dd111]
      -3.1537481925176414-10    -3.3164147721227693-10 I   flei

 -2 : -3.1500799889469005-10    -3.29915924109457  -10 I D0i[dd111]
      -3.150080001830792 -10    -3.2991592067243136-10 I   flei

 -1 : -3.112267506942415 -10    -3.135823319774082 -10 I D0i[dd111]
      -3.1122675069507063-10    -3.1358233197649007-10 I   flei
```