Feynman Integrals and Mellin-Barnes Representations

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- Introduction: Feynman integrals: $M$-point functions with $L$ loops and $N$ internal lines
- Loop momentum integrations with Feynman parameters $x_i$
- Doing the $x_i$-integrals
- Barnes’ contour integrals for the hypergeometric function
- Representations by Mellin-Barnes integrals (AMBRE package (K.Kajda))
- Treatment of divergencies in $d = 4 - 2\epsilon$ (MB package (M.Czakon))
- Numerical evaluations, infinite series, approximations (XSUMMER package (S.Moch, P.Uwer))
- Summary
Introductory remarks

For many problems of the past, a relatively simple approach to the evaluation of Feynman integrals was sufficient.

At most one-loop, at most $2 \rightarrow 2$ scattering (plus bremsstrahlung)

\[ T_{1l1m} = \frac{1}{\epsilon} + 1 + (1 + \frac{\zeta_2}{2})\epsilon + (1 + \frac{\zeta_2}{2} - \frac{\zeta_3}{3})\epsilon^2 + \]
\[ B_{4l2m} = \left[ -\frac{1}{\epsilon} + \ln(-s) \right] \frac{2y \ln(y)}{s(1-y^2)} + c_1\epsilon + \cdots \]

with $d = 4 - 2\epsilon$ and $m = 1$ and

\[ y = \frac{\sqrt{1-4/t-1}}{\sqrt{1-4/t+1}} \]

Figure shows so-called master integrals for Bhabha scattering (see lecture by R. Harlander for algebraic methods.)

Then Feynman parameters may be used and by direct integration over them one gets things like: $\frac{23}{57}, \ln \frac{t}{s}, \ln \frac{t}{s} \ln \frac{m^2}{s}, Li_2\left(\frac{t}{s}\right)$ etc. With more complexity of the reaction (more legs) and more perturbative accuracy (more loops), this approach appears to be not sufficiently sophisticated
Figure 1: Massive QED pentagon (5 variables), massless and massive hexagons (8 variables)

Figure 2: The two-loop planar QED box $B_1 = B_{7l4m1}$, another box master integral $B_{5l2m2}$ (from $B_2 = B_{7l2m2}$)
Figure 3: Two-loop box diagrams for massive Bhabha scattering

Figure 4: A box master integral $B_{5}l_{2}m_{2}$, related to $B_2 = B_{7}l_{2}m_{2}$ by shrinking two lines
Feynman integrals: scalar and tensor integrals, shrinked and dotted ones

Just to mention what kind of integrals may appear:

- diagrams with numerators:
  tensors arise from internal fermion lines: \( \int d^d k_i \cdots \frac{\gamma^\nu(k^\nu_i - p^\nu_n) - m_n}{(k_i - p_n)^2 - m_n^2} \cdots \)

- diagrams with shrinked and/or with dotted lines: a sample relation see next page

- relations to simpler diagrams may shift the complexity: \( \frac{1}{d-4} = -\frac{1}{2\epsilon} \)

\[
C_0(m, 0, m; m^2, m^2, s) = \frac{1}{s - 4m^2} \left[ \frac{d - 2}{d - 4} \frac{A_0(m^2)}{m^2} + \frac{2d - 3}{d - 4} B_0(m, m; s) \right]
\]

\[
V 3l2m = \frac{-1}{s - 4} \left[ \frac{1 - \epsilon}{\epsilon} T 1l1m + \frac{5 - 4\epsilon}{2\epsilon} SE 2l2m \right]
\]

(1)
\[
\text{SE312m}(a, b, c, d) = -e^{2\epsilon \gamma E} \frac{d^D k_1 d^D k_2 (k_1 k_2)^{-d}}{\pi^D} \int \frac{d^D k_1 d^D k_2 (k_1 k_2)^{-d}}{[(k_1 + k_2 - p)^2 - m^2]^b [k_1^2]^a [k_2^2 - m^2]^c}.
\]

\[
\text{SE312m} = \text{SE312m}(1, 1, 1, 0)
\]
\[
\text{SE312md} = \text{SE312m}(1, 1, 2, 0)
\]
\[
\text{SE312mN} = \text{SE312m}(1, 1, 1, -1)
\]

\[
\text{SE312md} = \frac{-(1+s) + \epsilon(2+s)}{s-4} \text{SE312m} + \frac{2(1-\epsilon)}{s-4} \left( \text{T11m}^2 + 3 \text{SE312mN} \right),
\]

\[
= \frac{1}{2\epsilon^2} + \frac{1}{2\epsilon} - \left( \frac{1 - \zeta_2}{2} \right) + \frac{1 + x}{1 - x} \ln(x) + \frac{1 + x^2}{(1 - x)^2} \frac{1}{2} \ln^2(x) + \mathcal{O}(\epsilon)
\]

Figure 5: The six two-loop 2-point masters of Bhabha scattering.
More legs, more loops

Seek methods for an (approximated) analytical or numerical evaluation of more involved diagrams

Remember: UltraViolet (UV) und InfraRed (IR) divergencies appear

Might try:

• same as for simple one-loop: Feynman parameters, direct integration

• use algebraic relations between integrals and find a (minimal) basis of master integrals – is a preparation of the final evaluation

• derive and solve (system of) differential equations

• derive and solve (system of) difference equations

• do something else for a direct evaluation of single integrals – of all of them or of the masters only

Into the last category falls what we present here:
Use Feynman parameters and transform the problem
Loop momentum integrations with Feynman parameters for $L$-loop $n$-point functions

Consider an arbitrary $L$-loop integral $G(X)$ with loop momenta $k_l$, with $E$ external legs with momenta $p_e$, and with $N$ internal lines with masses $m_i$ and propagators $1/D_i$,

$$G(X) = \frac{1}{(i\pi^{d/2})^L} \int \frac{d^D k_1 \ldots d^D k_L}{D_1^{\nu_1} \ldots D_i^{\nu_i} \ldots D_N^{\nu_N}} X(k_1, \ldots, k_L).$$

$$D_i = q_i^2 - m_i^2 = \left[ \sum_{l=1}^L c_l^i k_l + \sum_{e=1}^E d_i^e p_e \right]^2 - m_i^2$$

The numerator may contain a tensor structure

$$X = (k_1 p_{e_1}) \cdot \cdots \cdot (k_L p_{e_L}) = (k_1^{\alpha_1} \cdots k_L^{\alpha_L})(p_{e_1}^{\beta_1} \cdots p_{e_L}^{\beta_L})$$

Some numerators are reducible, i.e. one may divide them out against the numerators a la:

$$\frac{2kp_e}{[(k + p_e)^2 - m_1^2]D_2 \ldots D_N} \equiv \frac{[(k + p_e)^2 - m_1^2] - [k_2^2 - m_2^2] + (m_1^2 + m_2^2 - m_e^2)}{[(k + p_e)^2 - m_1^2]D_2 \ldots D_N}$$

$$= \frac{1}{D_2 \ldots D_N} - \frac{1}{[(k + p_e)^2 - m^2]D_3 \ldots D_N} + \frac{m_1^2 + m_2^2 - m_e^2}{D_1 D_2 \ldots D_N}$$
Irreducible numerators

For a two-loop QED box diagram, it is e.g. \( L = 2, E = 4 \), and we have as potential simplest numerators:

\[
k_1^2, k_2^2, k_1 k_2 \text{ and } 2(E - 1) \text{ products } k_1 p_e, k_2 p_e
\]

compared to \( N \) internal lines, \( N = 5, 6, 7 \). This gives

\[
I = L + L(L - 1)/2 + L(E - 1) - N \text{ irreducible numerators}
\]

of this type. Here:

\[
I(N) = 9 - N = 4, 3, 2
\]

This observation is of practical importance:

Imagine you search for potential masters. Then you may take into the list of masters at most (here e.g.) \( I(5) = 4 \), or \( I(6) = 3 \), or \( I(7) = 2 \) such integrals.

Which momenta combinations are irreducible is dependent on the choice of momenta flows.

Message:

When evaluating all Feynman integrals by Mellin-Barnes-integrals, one should also learn to handle numerator integrals

\[
\ldots \text{ and it is - in some cases - not too complicated compared to scalar ones}
\]

The one-loop case: \( L = 1, E = N \), so

\[
I(N) = 1 + (E - 1) - N = 0
\]

irreducible numerators
Introduce Feynman parameters

\[
\frac{1}{D_1^{\nu_1} D_2^{\nu_2} \ldots D_N^{\nu_N}} = \frac{\Gamma(\nu_1 + \ldots + \nu_N)}{\Gamma(\nu_1) \ldots \Gamma(\nu_N)} \int_0^1 dx_1 \ldots \int_0^1 dx_N \frac{x_1^{\nu_1-1} \ldots x_N^{\nu_N-1} \delta(1 - x_1 \ldots - x_N)}{(x_1 D_1 + \ldots + x_N D_N)^{N\nu}},
\]

with \( N_{\nu} = \nu_1 + \ldots + \nu_N \).

The denominator of \( G \) contains, after introduction of Feynman parameters \( x_i \), the momentum dependent function \( m^2 \) with index-exponent \( N_{\nu} \):

\[
(m^2)^{-\nu_1 + \ldots + \nu_N} = (x_1 D_1 + \ldots + x_N D_N)^{-N_{\nu}} = (k_i M_{ij} k_j - 2Q_j k_j + J)^{-N_{\nu}}
\]

Here \( M \) is an \((L \times L)\)-matrix, \( Q = Q(x_i, p_e) \) an \( L \)-vector and \( J = J(x_i x_j, m_i^2, p_{e_j} p_{e_l}) \).

\( M, Q, J \) are linear in \( x_i \). The momentum integration is now simple:

Shift the momenta \( k \) such that \( m^2 \) has no linear term in \( \vec{k} \):

\[
k = \vec{k} + (M^{-1})Q,
\]

\[
m^2 = \vec{k} M \vec{k} - Q M^{-1} Q + J.
\]

Remember:

\[
M^{-1} = \frac{1}{(\det M)} \tilde{M},
\]

where \( \tilde{M} \) is the transposed matrix to \( M \). The shift leaves the integral unchanged.
The shift leaves the integral unchanged (rename $\bar{k} \to k$):

$$I_k(1) = \int \frac{Dk_1 \ldots Dk_L}{(kMk + J - QM^{-1}Q)^{N_\nu}}.$$

Rotate now the $k^0 \to iK_E^0$ with $k^2 \to -k_E^2$ (and again rename $k^E \to k$):

$$I_k(1) \to (i)^L \int \frac{Dk_E^1 \ldots Dk_E^L}{(-k^E M k^E + J - QM^{-1}Q)^{N_\nu}} = (-1)^{N_\nu} (i)^L \int \frac{Dk_1 \ldots Dk_L}{[kMk - (J - QM^{-1}Q)]^{N_\nu}}.$$

Call

$$\mu^2(x) = -(J - QM^{-1}Q)$$

and get

$$I_k(1) = (-1)^{N_\nu} (i)^L \int \frac{Dk_1 \ldots Dk_L}{(kMk + \mu^2)^{N_\nu}}.$$

For 1-loop integrals - will use only those - we are ready. For L-loops go on and now diagonalize the matrix $M$ by a rotation:

$$k \to k'(x) = V(x) k,$$

$$kMk = k'M_{\text{diag}}k'$$

$$\rightarrow \sum \alpha_i(x)k_i^2(x),$$

$$M_{\text{diag}}(x) = (V^{-1})^+ M V^{-1} = (\alpha_1, \ldots, \alpha_L).$$
This leaves both the integration measure and the integral invariant:

\[ I_k(1) = (-1)^{N_{\nu}} (i)^L \int \frac{Dk_1 \ldots Dk_L}{(\sum_i \alpha_i k_i^2 + \mu^2)^{N_{\nu}}} \].

Rescale now the \( k_i \),

\[ \bar{k}_i = \sqrt{\alpha_i} k_i \]

with

\[ d^D k_i = (\alpha_i)^{-D/2} d^D \bar{k}_i, \]

\[ \prod_{i=1}^{L} \alpha_i = \det M, \]

and get the Euclidean integral to be calculated (and rename \( \bar{k} \rightarrow k \)):

\[ I_k(1) = (-1)^{N_{\nu}} (i)^L (\det M)^{-D/2} \int \frac{Dk_1 \ldots Dk_L}{(k_1^2 + \ldots + k_L^2 + \mu^2)^{N_{\nu}}}. \]

Use now (remembering that \( Dk = dk/(i\pi^{d/2}) \)):

\[ i^L \int \frac{Dk_1 \ldots Dk_L}{(k_1^2 + \ldots + k_L^2 + \mu^2)^{N_{\nu}}} = \frac{\Gamma \left( N_{\nu} - \frac{D}{2} L \right)}{\Gamma \left( N_{\nu} \right)} \frac{1}{(\mu^2)^{N_{\nu} - DL/2}}, \]  

(7)

\[ i^L \int \frac{Dk_1 \ldots Dk_L k_1^2}{(k_1^2 + \ldots + k_L^2 + \mu^2)^{N_{\nu}}} = \frac{d}{2} \frac{\Gamma \left( N_{\nu} - \frac{D}{2} L - 1 \right)}{\Gamma \left( N_{\nu} \right)} \frac{1}{(\mu^2)^{N_{\nu} - DL/2 - 1}}. \]

These formulae follow for \( L = 1 \) immediately from any textbook.

For \( L > 1 \), get it iteratively, with setting \((k_1^2 + k_2^2 + m^2)^N = (k_1^2 + M^2)^N, M^2 = k_2^2 + m^2, \) etc.
Finally, one gets for **Scalar integrals**:

\[
G(1) = (-1)^N \frac{\Gamma(N - D/2)}{\Gamma(1) \cdots \Gamma(N)} \int_0^1 \prod_{j=1}^N dx_j \ x_j^{\nu_j - 1} \delta \left( 1 - \sum_{i=1}^N x_i \right) \frac{(\det M)^{-D/2}}{(\mu^2)^{N - DL/2}},
\]

or

\[
G(1) = (-1)^N \frac{\Gamma(N - D/2)}{\Gamma(1) \cdots \Gamma(N)} \int_0^1 \prod_{j=1}^N dx_j \ x_j^{\nu_j - 1} \delta \left( 1 - \sum_{i=1}^N x_i \right) \frac{U(x)^{N - D(L+1)/2}}{F(x)^{N - DL/2}}
\]

with

\[
U(x) = (\det M) \ (\rightarrow 1 \text{ for } L = 1)
\]

\[
F(x) = (\det M) \mu^2 = -(\det M) \ J + Q \tilde{M} \ Q \ (\rightarrow -J + Q^2 \text{ for } L = 1)
\]

**Trick for one-loop functions:**

\[
U = \det M = 1 = \sum x_i \text{ and so } U \text{ ‘disappears’ and the construct } F_1(x) \text{ is bilinear in } x_i x_j:
\]

\[
F_1(x) = -J(\sum x_i) + Q^2 = \sum A_{ij} x_i x_j.
\]
The vector integral differs by some numerator $k_i p_e$ and thus there is a single shift in the integrand $k \to \vec{k} + U(x)^{-1} \tilde{M} Q$

the $\int d^d \vec{k} \frac{\bar{k}}{(\bar{k}^2 + \mu^2)} \to 0$, and no further changes:

$$G(k_1) = (-1)^{N_{\nu}} \frac{\Gamma(N_{\nu} - \frac{D}{2} L)}{\Gamma(\nu_1) \ldots \Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j \ x_j^{\nu_j - 1} \delta \left( 1 - \sum_{i=1}^N x_i \right) \frac{U(x)^{N_{\nu} - D(L+1)/2 - 1}}{F(x)^{N_{\nu} - DL/2}} \left[ \sum_l \tilde{M}_l Q_l \right] \alpha,$$

Here also a tensor integral:

$$G(k_1 k_2) = (-1)^{N_{\nu}} \frac{\Gamma(N_{\nu} - \frac{D}{2} L)}{\Gamma(\nu_1) \ldots \Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j \ x_j^{\nu_j - 1} \delta \left( 1 - \sum_{i=1}^N x_i \right) \frac{U(x)^{N_{\nu} - 2 - D(L+1)/2}}{F(x)^{N_{\nu} - DL/2}}$$

$$\times \sum_l \left[ [\tilde{M}_1 Q_l]_\alpha [\tilde{M}_2 Q_l]_\beta - \frac{\Gamma(N_{\nu} - \frac{D}{2} L - 1)}{\Gamma(N_{\nu} - \frac{D}{2} L)} \frac{g_{\alpha \beta}}{2} U(x) F(x) \left( V_{1l}^{-1} + V_{2l}^{-1} \right) \right].$$

The 1-loop case will be used in the following $L$ times for a sequential treatment of an $L$-loop integral (remember $\sum x_j D_j = k^2 - 2Qk + J$ and $F(x) = Q^2 - J$):

$$G([1, k p_e]) = (-1)^{N_{\nu}} \frac{\Gamma(N_{\nu} - \frac{D}{2})}{\Gamma(\nu_1) \ldots \Gamma(\nu_N)} \int_0^1 \prod_{j=1}^N dx_j \ x_j^{\nu_j - 1} \delta \left( 1 - \sum_{i=1}^N x_i \right) \frac{[1, Q p_e]}{F(x)^{N_{\nu} - D/2}}$$
Examples for one-loop $F$-polynomials

One-loop vertex:

$$F(t, m^2) = m^2(x_1 + x_2)^2 + [-t]x_1x_2$$

one-loop box:

$$F(s, t, m^2) = m^2(x_1 + x_2)^2 + [-t]x_1x_2 + [-s]x_3x_4$$

one-loop pentagon:

$$F(s, t, t', v_1, v_2, m^2) = m^2(x_1 + x_3 + x_4)^2 + [-t]x_1x_3 + [-t']x_1x_4 + [-s]x_2x_5 + [-v_1]x_3x_5 + [-v_2]x_2x_4$$

2-loop: B7l4m2, sub-loop with 2 off-shell legs (diagram see next page):

$$F^{-(a_{4567}-d/2)} = \left\{ [-t]x_4x_7 + [-s]x_5x_6 + m^2(x_5 + x_6)^2 ight. \\
\left. + (m^2 - Q_1^2)x_7(x_4 + 2x_5 + x_6) + (m^2 - Q_2^2)x_7x_5 \right\}^{-(a_{4567}-d/2)}$$

2-loop: B5l2m2, sub-loop with 2 off-shell legs (diagram see p.4):

$$F_{2lines}(k_1^2, m^2) = m^2(x_3)^2 + [-k_1^2 + m^2]x_1x_3$$
For both first subloops of one-loop type:

\[
K_{\text{up,1-loop}} = \frac{(-1)^{N_{\nu}} \Gamma(N_{\nu} - d/2)}{\prod \Gamma(\nu_i)} \int_0^1 \prod_j d x_j x_j^{\nu_j - 1} \frac{\delta(1 - \sum x_i)}{F^{N_{\nu} - d/2}}
\]

where \( N_{\nu} = \sum \nu_i \), and the second \( k \)-integral has to be done yet.

Figure 6: The planar 6- and 7-line topologies.
What to be done now?

Perform the $x$-integrations

Find an as-general-as-possible general formula

Make it ready for algorithmic analytical and/or numerical evaluation
Integrating the Feynman parameters – get MB-Integrals

In 2-loops, consider two subsequent sub-loops (the first: off-shell 1-loop, second on-shell 1-loop) and get e.g. for $B7l4m2$, the planar 2nd type 2-box:

$$K_{1\text{-loop Box,off}} = \frac{(-1)^{a_{4567}} \Gamma(a_{4567} - d/2)}{\Gamma(a_4)\Gamma(a_5)\Gamma(a_6)\Gamma(a_7)} \int_0^\infty \prod_{j=4}^7 dx_j x_j^{a_j-1} \delta(1 - x_4 - x_5 - x_6 - x_7) \frac{1}{F^{a_{4567}-d/2}}$$

where $a_{4567} = a_4 + a_5 + a_6 + a_7$ and the function $F$ is characteristic of the diagram; here for the off-shell 1-box (2nd type):

$$F^{-(a_{4567}-d/2)} = \left\{[-t]x_4x_7 + [-s]x_5x_6 + m^2(x_5 + x_6)^2 \right. \right.$$  
$$\left. + (m^2 - Q_{12}^2)x_7(x_4 + 2x_5 + x_6) + (m^2 - Q_{23}^2)x_7x_5 \right\}^{-(a_{4567}-d/2)}$$

We want to apply now:

$$\int_0^1 \prod_{j=4}^7 dx_j x_j^{a_j-1} \delta(1 - x_4 - x_5 - x_6 - x_7) = \frac{\Gamma(\alpha_4)\Gamma(\alpha_5)\Gamma(\alpha_6)\Gamma(\alpha_7)}{\Gamma(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7)}$$

with coefficients $\alpha_i$ dependent on $a_i$ and on $F$

See in a minute:
For this, we have to apply several MB-integrals here.
And do this, if needed, several times; here: repeat the procedure for the 2nd subloop.
\[
\int_0^1 \prod_{j=1}^N dx_j \ x_j^{\alpha_j-1} \delta \left(1 - \sum_{i=1}^N x_i \right) = \frac{\prod_{i=1}^N \Gamma(\alpha_i)}{\Gamma \left(\sum_{i=1}^N \alpha_i \right)}
\]

Simplest cases:

\[
\int_0^1 \ dx_1 \ x_1^{\alpha_1-1} \delta (1 - x_1) = 1
\]

\[
\int_0^1 \prod_{j=1}^2 dx_j \ x_j^{\alpha_j-1} \delta \left(1 - \sum_{i=1}^N x_i \right) = \int_0^1 dx_1 x_1^{\alpha_1-1} (1 - x_1)^{\alpha_2-1} = B(\alpha_1, \alpha_2) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)}
\]
Mathematical interlude

what we need is a sequential use of a formula like:

\[ \frac{1}{[A(x)+B(x)]^z} \Rightarrow A(x)^z A B(x)^z B \]

Remark:
Such a formula would also be useful in a completely different ideology:

\[ \frac{1}{(p^2-m^2)} \Rightarrow \frac{(m^2)^z m}{(p^2)^z p} \]

transforms a massive propagator into a massless propagator - with different exponent (index)
\[
\frac{1}{(A+B)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} d\zeta \Gamma(\lambda + \zeta) \Gamma(-\zeta) \frac{B^\zeta}{A^\lambda + \zeta}
\]
Mellin, Robert, Hjalmar, 1854-1933
Barnes, Ernest, William, 1874-1953
Barnes’ contour integrals for the hypergeometric function

Exact proof and further reading: Whittaker & Watson (CUP 1965) 14.5 - 14.52, pp. 286-290

Consider

\[ F(z) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} d\sigma (-z)^\sigma \frac{\Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(-\sigma)}{\Gamma(c+\sigma)} \]

where \(|\arg(-z)| < \pi\) (i.e. \((-z)\) is not on the neg. real axis) and the path is such that it separates the poles of \(\Gamma(a+\sigma)\Gamma(b+\sigma)\) from the poles of \(\Gamma(-\sigma)\).

1/\(\Gamma(c+\sigma)\) has no pole.

Assume \(a \neq -n\) and \(b \neq -n, n = 0, 1, 2, \cdots\) so that the contour can be drawn.

The poles of \(\Gamma(\sigma)\) are at \(\sigma = -n, n = 1, 2, \cdots\), and it is:

\[ \text{Residue}[ F[s] \Gamma[-s] , \{s,n\} ] = (-1)^n n! \ F(n) \]

Closing the path to the right gives then, by Cauchy’s theorem, for \(|z| < 1\) the
The hypergeometric function $\,_{2}F_{1}(a, b, c, z)$ (for proof see textbook):

$$
\frac{1}{2\pi i} \int_{-i\infty}^{+ i \infty} d\sigma (-z)^\sigma \frac{\Gamma(a+\sigma)\Gamma(b+\sigma)\Gamma(-\sigma)}{\Gamma(c+\sigma)} = \sum_{n=0}^{N \to \infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}
$$

$$
= \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} \,_{2}F_{1}(a, b, c, z)
$$

The continuation of the hypergeometric series for $|z| > 1$ is made using the intermediate formula

$$
F(z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(1-c+a+n)\sin[(c-a-n)\pi]}{\Gamma(1+n)\Gamma(1-a+b+n)\cos(n\pi)\sin[(b-a-n)\pi]} (-z)^{-a-n} 
$$

$$
+ \sum_{n=0}^{\infty} \frac{\Gamma(b+n)\Gamma(1-c+b+n)\sin[(c-b-n)\pi]}{\Gamma(1+n)\Gamma(1-a+b+n)\cos(n\pi)\sin[(a-b-n)\pi]} (-z)^{-b-n}
$$

and yields

$$
\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} \,_{2}F_{1}(a, b, c, z) = \frac{\Gamma(a)\Gamma(a-b)}{\Gamma(a-c)} (-z)^{-a} \,_{2}F_{1}(a, 1-c+a, 1-b+ac, z^{-1})
$$

$$
+ \frac{\Gamma(b)\Gamma(b-a)}{\Gamma(b-c)} (-z)^{-b} \,_{2}F_{1}(b, 1-c+b, 1-a+b, z^{-1})
$$
Corollary I

Putting \( b = c \), we see that

\[
2F_1(a, b, b, z) = \sum_{n=0}^{\infty} \frac{\Gamma(a + n)}{\Gamma(a)} \frac{z^n}{n!} = \frac{1}{(1 - z)^a} = \frac{1}{2\pi i} \frac{1}{\Gamma(a)} \int_{-i\infty}^{+i\infty} d\sigma \frac{(-z)^\sigma}{\Gamma(a + \sigma)\Gamma(-\sigma)}
\]

This allows to replace sum by product:

\[
\frac{1}{(A + B)^a} = \frac{1}{B^a[1 - (-A/B)]^a} = \frac{1}{2\pi i \Gamma(a)} \int_{-i\infty}^{+i\infty} d\sigma A^\sigma B^{-\sigma - a} \Gamma(a + \sigma)\Gamma(-\sigma)
\]
Barnes’ lemma

If the path of integration is curved so that the poles of $\Gamma(c - \sigma)\Gamma(d - \sigma)$ lie on the right of the path and the poles of $\Gamma(a + \sigma)\Gamma(b + \sigma)$ lie on the left, then

$$\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} d\sigma \Gamma(a + \sigma)\Gamma(b + \sigma)\Gamma(c - \sigma)\Gamma(d - \sigma) = \frac{\Gamma(a + c)\Gamma(a + d)\Gamma(b + c)\Gamma(b + d)}{\Gamma(a + b + c + d)}$$

It is supposed that $a, b, c, d$ are such that no pole of the first set coincides with any pole of the second set.

Scetch of proof: Close contour by semicircle $C$ to the right of imaginary axis. The integral exists and $\int_{C}$ vanishes when $\Re(a + b + c + d - 1) < 0$. Take sum of residues of the integrand at poles of $\Gamma(c - \sigma)\Gamma(d - \sigma)$. The double sum leads to two hypergeometric functions, expressible by ratios of $\Gamma$-functions, this in turn by combinations of $\sin$, may be simplifies finally to the r.h.s.

Analytical continuation: The relation is proved when $\Re(a + b + c + d - 1) < 0$. Both sides are analytical functions of e.g. $a$. So the relation remains true for all values of $a, b, c, d$ for which none of the poles of $\Gamma(a + \sigma)\Gamma(b + \sigma)$, as a function of $\sigma$, coincide with any of the poles of $\Gamma(c - \sigma)\Gamma(d - \sigma)$.

Corollary II Any real shift $k$: $\sigma + k, a - k, b - k, c + k, d + k$ together with $\int_{-k-i\infty}^{-k+i\infty}$ leaves the result true.
How can this be made useful in the context of Feynman integrals?

- Apply corollary I to propagators and get:

  \[
  \frac{1}{(p^2 - m^2)^a} = \frac{1}{2\pi i} \Gamma(a) \int_{-i\infty}^{i\infty} d\sigma \frac{(-m^2)\sigma}{(p^2)^{a+\sigma}} \Gamma(a + \sigma)\Gamma(-\sigma)
  \]

  which may allow to perform the (massless) momentum integral (with index \(a\) of the line changed to \((a + \sigma)\)).

- Apply corollary I after introduction of Feynman parameters and after the momentum integration to the resulting \(F\)- and \(U\)-forms, in order to get a single monomial in the \(x_i\), which allows the integration over the \(x_i\):

  \[
  \frac{1}{[A(s)x_1^{a_1} + B(s)x_1^{b_1}x_2^{b_2}]^a} = \frac{1}{2\pi i} \Gamma(a) \int_{-i\infty}^{i\infty} d\sigma [A(s)x_1^{a_1}]^\sigma [B(s)x_1^{b_1}x_2^{b_2}]^{a+\sigma} \Gamma(a + \sigma)\Gamma(-\sigma)
  \]

  Both methods leave Mellin-Barnes (MB-) integrals to be performed afterwards.
A short remark on history

- **N. Usyukina, 1975**: "ON A REPRESENTATION FOR THREE POINT FUNCTION", Teor. Mat. Fiz. 22;
a finite massless off-shell 3-point 1-loop function represented by 2-dimensional MB-integral

N-point 1-loop functions represented by n-dimensional MB-integral

- **V. Smirnov, 1999**: "Analytical result for dimensionally regularized massless on-shell double box", Phys. Lett. B460 (1999);
treat UV and IR divergencies by analytical continuation: shifting contours and taking residues 'in an appropriate way'

- **B. Tausk, 1999**: "Non-planar massless two-loop Feynman diagrams with four on-shell legs", Phys. Lett. B469 (1999);
nice algorithmic approach to that, starting from search for some unphysical space-time dimension $d$ for which the MB-integral is finite and well-defined

Tausk’s approach realized in Mathematica program MB.m, published and available for use
The $\Gamma$-function

The $\Gamma$-function may be defined by a difference equation:

$$z\Gamma(z) - \Gamma(z + 1) = 0$$

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$$

Series[\text{Gamma}[\text{ep}], \{\text{ep, 0, 2}\}] =

$$\Gamma[\epsilon] = \frac{1}{\epsilon} - \gamma_E + \frac{1}{12} (6\gamma_E^2 + \pi^2)\epsilon + \frac{1}{12} (-2\gamma_E^3 - \gamma_E^2\pi + 2\Psi(2, 1))\epsilon^2 + \cdots$$

exp(\text{ep EulerGamma})\text{Series[Gamma[ep], \{ep, 0, 2\}] =}

$$e^{\epsilon\gamma_E} \Gamma[\epsilon] = \frac{1}{\epsilon} + \frac{1}{12} (-\epsilon^2) + \frac{1}{6} (\Psi(2, 1))\epsilon^2 + \cdots$$
Singularities in the complex plane:

J. Gluza and T. Riemann, 29/30 March 2007 - CAPP, Zeuthen
Some facts on residua

The function

\[ F(z) = \sum_{i=-N}^{\infty} \frac{a_i}{(z-z_0)^i} \]

has the residue

\[ \text{Res } F(z)|_{z=z_0} = a_{-1} \]

An integral over an anti-clockwise directed closed path \( C \) around \( z_0 \) then is

\[ \frac{1}{2\pi i} \int_C dz F(z) = 2\pi i a_{-1} \]

If \( G(z) \) has a Taylor expansion around \( z_0 \) and \( F(z) \) has a Laurent expansion beginning with \( a_{-N}/(z-z_0)^N + \ldots \), then their product has the residue:

\[ \text{Res}[G(z) F(z)]|_{z=z_0} = \sum_{k=1}^{N} \frac{a_{-k} G(z_0)^{(k)}}{k!} \]
Some resida with $\Gamma(z)$ and $\Psi(z)$

Residue[$F[z]\Gamma[z], \{z, -n\}] = \frac{(-1)^n}{n!} F[-n]

Residue[$F[z]\Gamma[z]^2, \{z, -n\}] = \frac{2 PolyGamma[n+1]F[-n]+F'[-n]}{(n!)^2}$

Residue[$F[z]\Gamma[z - 1]^2, \{z, -3\}] = \frac{25F[-3]-12\gamma E F[-3]+6F'[-3]}{3456}$

Series[$F[z]\Gamma[z - 1]^2, \{z, -3, -1\}] = \frac{F[-3]}{576(z+3)^2} + \frac{25F[-3]-12\gamma E F[-3]+6F'[-3]}{3456(z+3)}$

$\quad + a_0 + a_1(z + 3) + \cdots$

Series[$\Gamma[z + a]\Gamma[z - 1]^2, \{z, -3, -1\}] = \frac{\Gamma[-3+a]}{576(z+3)^2}$

$\quad + \frac{(25\Gamma[-3+a]-12\gamma E \Gamma[-3+a]+6(\Gamma[-3+a] PolyGamma[0,-3+a])}{3456}$

$\quad + a_0 + a_1(z + 3) + \cdots$
Where

\[ \text{Polygamma}[n + 1] \equiv \text{Polygamma}[0, n + 1] \]

\[ = \Psi(n + 1) = \frac{\Gamma'(n + 1)}{\Gamma(n + 1)} = S_1(n) - \gamma_E = \sum_{k=1}^{n} \frac{1}{k} - \gamma_E \]

The following properties hold:

\[ \Psi(z + 1) = \Psi(z) + \frac{1}{z} \quad (8) \]
\[ \Psi(1 + \epsilon) = -\gamma_E + \zeta_2 \epsilon + \ldots \quad (9) \]
\[ \Psi(1) = -\gamma_E \quad (10) \]
\[ \Psi(2) = 1 - \gamma_E \quad (11) \]
\[ \Psi(3) = 3/2 - \gamma_E \]
Some sums Mathematica can do

\[ \text{Sum}[s^n \Gamma(n+1)^3/(n!\Gamma(2+2n)), n, 0, \text{Infinity}] = (4\text{ArcSin[Sqrt[s]/2]}/(\text{Sqrt}[4 - s]*\text{Sqrt}[s])) \]

\[ \text{Sum}[s^n \text{PolyGamma}[0, n+1], n, 0, \text{Infinity}] = (\text{EulerGamma} + \text{Log}[1 - s])/(1 - s) \]
The Feynman integral $V3l2m$ is the QED one-loop vertex function, which is no master. It is infrared-divergent (see this by counting of powers of loop integration momentum $k$ or know it from: massless line between two external on-shell lines)

$$F = m^2 (x_1 + x_2)^2 + [-s] x_1 x_2$$

Here: $s \equiv t$ (sorry!!). We will also use the variable

$$y = \frac{\sqrt{-s + 4} - \sqrt{-s}}{\sqrt{-s + 4} + \sqrt{-s}}$$

$$V3l2m[y] = \frac{e^{\epsilon\gamma_E} \Gamma(-2\epsilon)}{2\pi i} \int dz (-s)^{-\epsilon-1-z} \frac{\Gamma^2(-\epsilon-z)\Gamma(-z)\Gamma(1+\epsilon+z)}{\Gamma(1-2\epsilon)\Gamma(-2\epsilon-2z)}$$

$$= \frac{V3l2m[-1,y]}{\epsilon} + V3l2m[0,y] + \epsilon V3l2m[1,y] + \cdots.$$  \hspace{1cm} (12)
\[ V_{312m}[-1, y] = \frac{1}{2} \frac{1}{2\pi i} \int_{-i\infty + u}^{+i\infty + u} dr (-t)^{-1-r} \frac{\Gamma^3[-r]\Gamma[1+r]}{\Gamma[-2r]} \]

\[ = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(t)^n}{\binom{2n}{n}(2n+1)} \]

\[ = \frac{1}{2} \frac{4 \arcsin(\sqrt{t/2})}{\sqrt{4-t\sqrt{t}}} \]

\[ = \frac{1}{2} \frac{-2y(t)}{1 - y^2(t)} \ln y(t) \]

(13)

Close path upwards to the left, so the infinite series of residua of

\[ \Gamma[1+r] \]

at \( r = -n, n = 1, 2, \cdots \) arises with weight function

\[ G(r) = (-t)^{-1-r} \frac{\Gamma^3[-r]}{\Gamma[-2r]} \]

and the sum may be done with Mathematica, see p.33.
\[ V312m[0, y] = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dr \frac{\Gamma^3[-r] \Gamma[1 + r]}{\Gamma[-2r]} \sum_{n=0}^{\infty} \frac{t^n}{\binom{2n}{n}} \frac{S_1(n)}{(2n + 1)} \]

\[ = \frac{1}{2} \left[ \gamma_E - \ln(-s) + 2\Psi[-2r] - 2\Psi[-r] + \Psi[1 + r] \right] \tag{14} \]
and

\[
V312m[1, y] = \frac{1}{4} \frac{1}{2\pi i} \int_{-i\infty + u}^{+i\infty + u} dr(-t)^{-1-r} \frac{\Gamma^3(-r)\Gamma[1 + r]}{\Gamma[-2r]} \left[ \gamma_E^2 + \text{Log}[-s]^2 + \text{Log}[-s](-2\gamma_E - 4\Psi[-2z] + 4\Psi[z] - 2\Psi[1 + z]) \\
+ \gamma_E(4\Psi[-2z] - 4\Psi[z] + 2\Psi[1 + z]) \\
- 4\Psi[1, -2z] + 2\Psi[1, -z] + \Psi[1, 1 + z] \\
+ 4(\Psi[-2z]^2 - 2\Psi[-2z]\Psi[-z] + \Psi[-z]^2 + \Psi[-2z]\Psi[1 + z] \\
- \Psi[-z]\Psi[1 + z]) + \Psi[1 + z]^2 \right] \\
= \text{const} \frac{1}{4} \sum_{n=0}^{\infty} \frac{(t)^n}{(2n)\binom{2n}{n}(2n + 1)} \left[ S_1(n)^2 + \zeta_2 - S_2(n) \right]. \tag{15}
\]

Here, \( \Psi[r] = \ldots \) and \( \Psi[1, r] = \ldots \), and the harmonic numbers \( S_k(n) \) are

\[
S_k(n) = \sum_{i=1}^{n} \frac{1}{i^k},
\]

see e.g. talk by S.Moch.
Experimentally,

\[ V312m[2, y] = \frac{1/12}{2\pi i} \int_{-i\infty + u}^{+i\infty + u} dz(-t)^{-1-z} \frac{\Gamma^3[-z]\Gamma[1 + z]}{\Gamma[-2z]} \left[ a(z) + c_1(z)\Psi(0, 1 + z) + \Psi(2, 1 + z) + 2\Psi(0, 1 + z)^2 + \Psi(0, 1 + z)^3 + 3\Psi(0, 1 + z)\Psi(1, 1 + z) + d_1(z)[\Psi(1, 1 + z) + 2\Psi(0, 1 + z)^2] \right] \] (16)

with some longer coefficients \(cc1\), \(d_1(z)\), \(aa\):

\[ cc1 = 3*EulerGamma^2 - 6*EulerGamma*Log[-s] + 3*Log[-s]^2 + 12*PolyGamma[0, -2*z]^2 + 6*PolyGamma[0, -2*z]*(2*(EulerGamma - Log[-s]) - 4*PolyGamma[0, -z]) - 12*(EulerGamma - Log[-s])*PolyGamma[0, -z] + 12*PolyGamma[0, -z]^2 - 12*PolyGamma[1, -2*z] + 6*PolyGamma[1, -z] \]

and

\[ d_1(z) = 3*EulerGamma - 3*Log[-s] + 6*PolyGamma[0, -2*z] - 6*PolyGamma[0, -z] \] (17)
Finally,

\[
aa = 
\begin{align*}
&\text{EulerGamma}^3 - 3\text{EulerGamma}^2\text{Log}[-s] + 3\text{EulerGamma}\text{Log}[-s]^2 \\
&\quad - \text{Log}[-s]^3 + 8\text{PolyGamma}[0, -2z]^3 \\
&\quad + 12\text{PolyGamma}[0, -2z]^2(\text{EulerGamma} - \text{Log}[-s] - 2\text{PolyGamma}[0, -z]) \\
&\quad + 12(\text{EulerGamma} - \text{Log}[-s])\text{PolyGamma}[0, -z]^2 \\
&\quad - 8\text{PolyGamma}[0, -z]^3 - 12\text{EulerGamma}\text{PolyGamma}[1, -2z] \\
&\quad + 12\text{Log}[-s]\text{PolyGamma}[1, -2z] \\
&\quad + 6\text{EulerGamma}\text{PolyGamma}[1, -z] - 6\text{Log}[-s]\text{PolyGamma}[1, -z] \\
&\quad - 6\text{PolyGamma}[0, -z](\text{EulerGamma}^2 - 2\text{EulerGamma}\text{Log}[-s] + \text{Log}[-s]^2) \\
&\quad - 4\text{PolyGamma}[1, -2z] + 2\text{PolyGamma}[1, -z]) \\
&\quad + 6\text{PolyGamma}[0, -2z](\text{EulerGamma}^2 - 2\text{EulerGamma}\text{Log}[-s] + \text{Log}[-s]^2) \\
&\quad - 4(\text{EulerGamma} - \text{Log}[-s])\text{PolyGamma}[0, -z] \\
&\quad + 4\text{PolyGamma}[0, -z]^2 - 4\text{PolyGamma}[1, -2z] + 2\text{PolyGamma}[1, -z]) \\
&\quad + 8\text{PolyGamma}[2, -2z] - 2\text{PolyGamma}[2, -z]
\end{align*}
\]
\[ V_{312m}[2, y] = \sum_{n=0}^{\infty} \frac{s^n}{\binom{2n}{n}} \left( \frac{2n+1}{(2n+1)} \right) \left[ \frac{1}{12} S_1[n]^3 - \frac{1}{4} S_1[n] S_2[n] + \frac{1}{4} \zeta_2 S_1[n] + \frac{1}{6} S_3[n] - \frac{1}{6} \zeta_3 \right]. \quad (18) \]

Sum this up!!

Answer is known to us from another technique: differential equations; see our Bhabha webpage, file master.m
On-shell example: B4l2m, the 1-loop on-shell box

den = (x4 d4 + x5 d5 + x6 d6 + x7 d7 // Expand) /. kinBha /. m^2 -> 1 // Expand

Q = -Coefficient[den, k]/2 // Simplify
   = p3 x4 + p2 x5 - p1 (x4 + x6)

M = Coefficient[den, k^2] // Simplify
   = x4 + x5 + x6 + x7 -> 1

J = den /. k -> 0 // Simplify
   = t x4

F[x] = (Q^2 - J M // Expand) /. kinBha /. m^2 -> 1 /. u -> -s - t + 4 // Expand
   = (x5+x6)^2 + (-s)x5x6 + (-t)x4x7

B4l2ma = mb[(x5+x6)^2, -tx7x4 - sx5x6, nu, ga]

B4l2mb = B4l2ma /. (-sx5x6 - tx4x7)^(-ga - nu) ->
   mb[(-s)x5x6, (-t)x7x4, nu+ga, de]
   /./((-s)x5x6)^de_ -> (-s)^de x5^de x6^de
   /./(x56^2)^ga -> (x5 + x6)^(2ga)
\[
(\text{inv2piI}^2(-s)^{\text{de}} x5^{\text{de}} x6^{\text{de}} ((x5 + x6)^{(2\text{ga})}((-t)x4x7)^{(-\text{de}-\alpha)\text{nu}})
\]
\[
\Gamma[-\text{de}] \Gamma[-\alpha] \Gamma[\text{de} + \alpha + \text{nu}] /\Gamma[\text{nu}]
\]

\[
\text{B412mc} = \text{B412mb} /. (x5 + x6)^{(2\alpha)} \rightarrow \\
\text{mb}[x5, x6, -2\alpha, \alpha] \\
/. ((-t)x4x7)^{\alpha} \rightarrow (-t)^{\alpha} x4^{\alpha} x7^{\alpha} // \text{ExpandAll}
\]

\[
= 1/(\Gamma[-2\alpha] \Gamma[\text{nu}])
\]
\[
\text{inv2piI}^3 (-s)^{\text{de}} (-t)^{(-\text{de}-\alpha-\text{nu})}
\]
\[
x4^{(-\text{de}-\alpha-\text{nu})} x5^{(\text{de}+\alpha)} x6^{(\text{de}+2\alpha-\alpha)} x7^{(-\text{de}-\alpha-\text{nu})}
\]
\[
\Gamma[-\text{de}] \Gamma[-\alpha] \Gamma[\text{de} + \alpha + \text{nu}] \Gamma[-\alpha] \Gamma[-2\alpha + \alpha]
\]

\[
\text{B412md} = \text{xfactor4}[a4, x4, a5, x5, a6, x6, a7, x7] \text{B412mc}
\]

\[
= \ldots (-s)^{\text{de}} (-t)^{(-\text{de}-\alpha-\text{nu})}
\]
\[
x4^{(-1+a4-\text{de}-\alpha-\text{nu})} x5^{(-1+a5+\text{de}+\alpha)} x6^{(-1+a6+\text{de}+2\alpha-\alpha)} x7^{(-1+a7- \text{de}-\alpha-\text{nu})}
\]

\[
\text{B412me} = \\
\text{B412md} /. \\
x4^{\text{B4} -} x5^{\text{B5} -} x6^{\text{B6} -} x7^{\text{B7} -} \rightarrow \text{xint4}[x4^{\text{B4}} x5^{\text{B5}} x6^{\text{B6}} x7^{\text{B7}}]
\]

\[
= \ldots
\]
B4l2mf = B4l2me /.
   Gamma[a6 + de + 2 ga - si]Gamma[-si]Gamma[ a5 + de + si] Gamma[-2 ga + si]
   -> barne1[si, a5 + de, -2 ga, a6 + de + 2 ga, 0]

This finishes the evaluation of the MB-representation for B4l2m.
Package: AMBRE.m (K. Kajda, with support by J. Gluza and TR)
B4l2m

\[ F[x] = (Q^2 - J M // \text{Expand}) /. \text{kinBha} /. m^2 -> 1 /. u -> -s - t + 4 // \text{Expand} \]
\[ = (x5+x6)^2 + (-s)x5x6 + (-t)x4x7 \]

B4l2m, the 1-loop QED box, with two photons in the s-channel; the Mellin-Barnes representation reads for finite \( \epsilon \):

\[ \text{B4l2m} = \text{Box}(t, s) = e^{\epsilon \gamma_E} \frac{1}{\Gamma[-2\epsilon](-t)^{(2+\epsilon)}} \frac{1}{(2\pi i)^2} \int_{-i\infty}^{+i\infty} dz_1 \int_{-i\infty}^{+i\infty} dz_2 \]
\[ \frac{(-s)^{z_1}(m^2)^{z_2}}{(-t)^{z_1+z_2}} \Gamma[2 + \epsilon + z_1 + z_2] \Gamma[2][1 + z_1] \Gamma[-z_1] \Gamma[-z_2] \]
\[ \Gamma^2[-1 - \epsilon - z_1 - z_2] \frac{\Gamma[-2 - 2\epsilon - 2z_1]}{\Gamma[-2 - 2\epsilon - 2z_1 - 2z_2]} \]

(19)

Mathematica package MB used for analytical expansion \( \epsilon \rightarrow 0 \): [Czakon:2005rk]
$$B4l2m = -\frac{1}{\epsilon} I1 + \ln(-s) I1 + \epsilon \left( \frac{1}{2} \left[ \zeta(2) - \ln^2(-s) \right] I1 - 2I2 \right). \quad (20)$$

with $I1$ being also the divergent part of the vertex function $C0(t; m, 0, m)/s = V3l2m/s$ (as is well-known):

$$I1 = \frac{e^{\epsilon E}}{st} \frac{1}{2\pi i} \int_{-\frac{1}{2} - i\infty}^{-\frac{1}{2} + i\infty} dz1 \left( \frac{m^2}{-t} \right)^{z1} \frac{\Gamma^3[-z1]\Gamma[1 + z1]}{\Gamma[-2z1]} = \frac{1}{m^2s} \frac{2y}{1 - y^2} \ln(y) \quad (21)$$

with $y = (\sqrt{1 - 4m^2}/t - 1)/(\sqrt{1 - 4m^2}/t + 1)$: close contour to left, take residua at $(1 + z1) = -n$, sum up with Mathematica:

$$\text{Residue}[F[x] \Gamma[1 + x], \{x, -n\}] // \text{InputForm} = -(-1)^n F[-n]/n!$$

$$\text{Sum}[s^n \Gamma[n + 1]^3/(n! \Gamma[2 + 2n]), \{n, 0, \infty\}] // \text{InputForm} = (4 \text{ArcSin}[Sqrt[s]/2])/(Sqrt[4 - s]*Sqrt[s])$$

The $I2$ is more complicated:

$$I2 = \frac{e^{\epsilon E}}{t^2} \frac{1}{(2\pi i)^2} \int_{-\frac{3}{4} - i\infty}^{-\frac{3}{4} + i\infty} dz1 \left( \frac{-s}{-t} \right)^{z1} \Gamma[-z1] \Gamma[-2(1 + z1)] \Gamma^2[1 + z1] \times \int_{-\frac{1}{2} - i\infty}^{-\frac{1}{2} + i\infty} dz2 \left( \frac{m^2}{-t} \right)^{z2} \frac{\Gamma[-z2] \Gamma^2[-1 - z1 - z2]}{\Gamma[-2(1 + z1 + z2)]} \Gamma[2 + z1 + z2]. \quad (22)$$
The expansion of $\text{B412m}$ at small $m^2$ and fixed value of $t$

With

$$m_t = \frac{-m^2}{t}, \quad (23)$$

$$r = \frac{s}{t}, \quad (24)$$

Look, under the integral, at $(-m^2/t)^{z_2}$, and close the path to the right.

Seek the residua from the poles of $\Gamma$-functions with the smallest powers in $m^2$ and try to sum the resulting series.

Automatize this, it is not too easy.

we have obtained a compact answer for $I_2$ with the additional aid of XSUMMER $[\text{Moch:2005uc}]$.

The box contribution of order $\epsilon$ in this limit becomes:

$$\text{B412m}[t, s, m^2; +1] = \frac{1}{st} \left\{ 4\zeta_3 - 9\zeta_2 \ln(m_t) + \frac{2}{3} \ln^3(m_t) + 6\zeta_2 \ln(r) - \ln^2(m_t) \ln(r) \right. \right.$$  

$$\left. + \frac{1}{3} \ln^3(r) - 6\zeta_2 \ln(1 + r) + 2\ln(-r) \ln(r) \ln(1 + r) - \ln^2(r) \ln(1 + r) \right. \right.$$  

$$\left. + 2\ln(r) \text{Li}_2(1 + r) + 2\text{Li}_3(-r) \right\} + \mathcal{O}(m_t). \quad (25)$$
Some routines in mathematica which were used:

(* Barnes' first lemma: \[ \int d(si) \Gamma(si1p+si)\Gamma(si2p+si)\Gamma(si1m-si)\Gamma(si2m-si) \]
with \[ 1/inv2piI = 2 \pi i \]*)

barne1[si_, si1p_, si2p_, si1m_, si2m_] :=
    1/inv2piI Gamma[si1p + si1m] Gamma[si1p + si2m] Gamma[
        si2p + si1m] Gamma[si2p + si2m] /Gamma[si1p + si2p + si1m + si2m]

(* Mellin-Barnes integral: \((A+B)^{-nu} = 1/(2 \pi i) \int d(si) a^si b^{(-nu - si)} \Gamma[-si]\Gamma[nu+si]/\Gamma[nu] \]*)

mb[a_, b_, nu_, si_]:=inv2piI a^si b^{(-nu-si)}Gamma[-si]Gamma[nu+si]/Gamma[nu]

(* After the k-integration, the integrand for \[ \int \prod dxi xi^{(ai - 1)} \delta(1-\sum xi) \]
will be (L=1 loop) : xfactor F^{(-nu)} Q(xi).pe with \[ nu = a1 + .. + an - d/2 \]*)

xfactor3[a1_, x1_, a2_, x2_, a3_, x3_] :=
    I Pi^{(d/2)} (-1)^{(a1 + a2 + a3)} x1^{(a1 - 1)} x2^{(a2 - 1)} x3^{(a3 - 1)}Gamma[
        a1 + a2 + a3 - d/2]/(Gamma[a1]Gamma[a2]Gamma[a3])

(* xinti - the i-dimensional x - integration over Feynman parameters /16 06 2005 \]*)

xint3[x1_^(a1_) x2_^(a2_) x3_^(a3_) ] :=
    Gamma[a1 + 1] Gamma[a2 + 1] Gamma[a3 + 1] / Gamma[a1 + a2 + a3 + 3]
Another nice box with numerator, $\mathbf{B5l3m}(p_e \cdot k_1)$

We used it for the determination if the small mass expansion.

\[
\mathbf{B5l3m}(p_e \cdot k_1) = \frac{m^4 (-1)^{a_{12345}} e^{2 \epsilon \gamma E}}{\prod_{j=1}^5 \Gamma[a_i] \Gamma[5 - 2 \epsilon - a_{123}] (2\pi i)^4} \int_{-i\infty}^{+i\infty} d\alpha \int_{-i\infty}^{+i\infty} d\beta \int_{-i\infty}^{+i\infty} d\gamma \int_{-i\infty}^{+i\infty} d\delta \\
(\gamma - s) (4 - 2 \epsilon - a_{12345} - \alpha - \beta - \delta) (-t)^{\delta} \\
\frac{\Gamma[6 - 3 \epsilon - a_{12345} - \alpha]}{\Gamma[2 - \epsilon - a_{13} - \alpha - \gamma]} \frac{\Gamma[7 - 3 \epsilon - a_{12345} - \alpha]}{\Gamma[5 - 2 \epsilon - a_{123}]} \frac{\Gamma[4 - 2 \epsilon - a_{1123} - 2 \alpha - \gamma]}{\Gamma[5 - 2 \epsilon - a_{1123} - 2 \alpha - \gamma]} \frac{\Gamma[4 - 2 \epsilon - a_{1234} - \alpha - \beta - \delta]}{\Gamma[5 - 2 \epsilon - a_{1234} - \alpha - \beta - \delta - \gamma]}
\]
This kind of expression now has to be evaluated:

- Check special cases of indices, set lines to 1 (by setting \( a_i \to 0 \) if possible)
- Extract the \( \epsilon \)-dependence related to UV and IR singularities (see next pages)
- After that: may set \( s < 0, \ t < 0 \) and evaluate numerically Euclidean case
- Use sector decomposition for a numerical comparison - if you have a program for that
- Try to go Minkowskian in a numerical way (if you like this)
- Go on analytically, e.g. by taking residua \( \to \) get nested infinite sums from the residua
- Try to sum them up
Shrinking of lines; seek the $\epsilon$-expansion

Go on with some study of the 2nd planar 2-box, B7l4m2 (see also Smirnov book 4.73):

$$B_{pl,2} = \frac{\text{const}}{(2\pi i)^6} \int_{-i\infty}^{+i\infty} \frac{m_\mathcal{M}^{z_5+z_6}}{-s} \left[ \frac{-t}{-s} \right]^{z_1} \prod_{j=1}^{6} [dz_j \Gamma(-z_j)] \prod_{k=7}^{18} \Gamma_k(\{z_i\}) \prod_{l=19}^{24} \Gamma_l(\{z_i\})$$

with $a = a_1 + \ldots + a_7$ and

$$z_i = \text{const} + i \Im m(z_i)$$
$$d = 4 - 2\epsilon$$
$$\text{const} = \frac{(i\pi^{d/2}/2)^2 (-1)^a (-s)^{d-a}}{\Gamma(a_2)\Gamma(a_4)\Gamma(a_5)\Gamma(a_6)\Gamma(a_7)\Gamma(d - a_{4567})}$$

The integrand includes e.g.:

$$\Gamma_2 = \Gamma(-z_2)$$
$$\Gamma_4 = \Gamma(-z_4)$$
$$\Gamma_7 = \Gamma(a_4 + z_2 + z_4)$$
$$\Gamma_8 = \Gamma(D - a_{445667} - z_2 - z_3 - 2z_4)$$
$$...$$
Figure 7: The planar 6- and 7-line topologies.

Figure 8: The 5-line topologies. **B7l4m2**: shrink line 1 get **B6l3m2**, then line 4 get **B5l3m**
Example:
derive from B7l4m2 the MB-integral for B5l3m by setting $a_1 = 0$ (trivial, gives B6l3m2) and then setting $a_4 = 0$.
The latter do with care because of
\[ \frac{1}{\Gamma(a_4)} \to \frac{1}{\Gamma(0)} = 0 \]

See by inspection that we will get factor $\Gamma(a_4)$ if $z_2, z_4 \to 0$.

→ Start with the $z_2, z_4$ integrations by
taking the residues for closing the integration contours to the right:
\[ I_{2,4} = \frac{(-1)^2}{(2\pi i)^2} \int dz_2 \Gamma(-z_2) \int dz_4 \frac{\Gamma(a_4 + z_2 + z_4)}{\Gamma(a_4)} \Gamma(-z_4) R(z_i) \]
\[ = \frac{1}{(2\pi i)^2} \int dz_2 \Gamma(-z_2) \sum_{n=0,1,\ldots} \frac{(-1)^n}{n!} \frac{\Gamma(a_4 + z_2 + n)}{\Gamma(a_4)} R(z_i) \]
\[ = \sum_{n,m=0,1,\ldots} \frac{(-1)^{n+m}}{n! m!} \frac{\Gamma(a_4 + n + m)}{\Gamma(a_4)} R(z_i) \to a_4 = 0 \quad 1 \times R(z_i) \]

So, setting $a_1 = a_4 = 0$ and eliminating $\int dz_2 dz_4$ with setting $z_2 = z_4 = 0$
we got a 4-fold Mellin-Barnes integral for topology B5l3m (by "shrinking of lines")
with $24 - 3 = 21$ $z_i$-dependent $\Gamma$-functions which may yield residua within four-fold sums.
The MB-representation has to be calculated explicitly at fixed indices, e.g.

\[ B_{5l3md2} = \frac{B_2}{\epsilon^2} + \frac{B_1}{\epsilon} + B_0 \]

**General Tasks, first two steps automated by MB.m:**

- Find a region of definiteness of the n-fold MB-integral

  \[ \Re(z_1) = -1/80, \Re(z_3) = -33/40, \Re(z_5) = -21/20, \Re(z_6) = -59/160, \Re(\epsilon) = -1/10! \]

- Then go to the physical region where \( \epsilon \ll 1 \) by distorting the integration path step by step (adding each crossed residuum – per residue this means one integral less!!!)

- Take integrals by sums over residua, i.e. introduce infinite sums

- Sum these infinite multiple series into some known functions of a given class, e.g. Nielsen polylogs, Harmonic polylogs or whatever is appropriate.
An important tool is the command FindInstance of Mathematica 5: It allows to solve a system of inequalities.

Here an example for B7l4m3, the non-planar massive double box:

```mathematica
sol = FindInstance[
    Cases[B7l4m3 ... Gamma[x_] -> x > 0 /. ep -> -1/10, {z1, z2, z3, z4, z5, z6, z7, z8}]
]
```

The result is:

```
{z1 -> -1/20, z2 -> -1/40, z3 -> -1/20, z4 -> -29/32, 
z5 -> -67/80, z6 -> -83/160, z7 -> -273/320, z8 -> -5/64}
```

Really, all arguments are positive:

```
```

Now set \( \epsilon = 0 \):

```
{z1 -> -1/20, z2 -> -1/40, z3 -> -1/20, z4 -> -29/32, 
z5 -> -67/80, z6 -> -83/160, z7 -> -273/320, z8 -> -5/64, ep -> 0}
```

Determine again the arguments of the Gamma-functions; observe:

2 arguments are negative now: those for G3 and G8

```
```

Perform the corresponding shifts of integration curve, add the residua and again perform the test for the arguments of the new, lower-dimensional MB-integrals.
We derived an algorithmic solution for isolating the singularities in $1/\epsilon$

The automatization of that: $MB.m$ (M. Czakon)

\[ B5l3md2 \rightarrow MB(4\text{-dim, fin}) + MB_3(3\text{-dim, fin}) \]
\[ + MB_{36}(2\text{-dim, } \epsilon^{-1}, \text{fin}) + MB_{365}(1\text{-dim, } \epsilon^{-2}, \epsilon^{-1}, \text{fin}) \]
\[ + MB_5(3\text{-dim, fin}) \]

After these preparations e.g.:

\[ MB_{365}(1\text{-dim, } \epsilon^{-2}) \sim \frac{1}{\epsilon^2} \frac{1}{2\pi i} \int dz_6 \frac{(-s)^{-z_6-1} \Gamma(-z_6)^3 \Gamma(1 + z_6)}{8 \Gamma(-2z_6)} \]
\[ = \frac{1}{\epsilon^2} \sum_{n=0,\infty} (-1)^n (-s)^n \Gamma(1 + n)^3 \frac{1}{8n! \Gamma(-2(-1 - n))} \]
\[ = -\frac{1}{\epsilon^2} \frac{ArcSin(\sqrt{s}/2)}{2\sqrt{4 - s\sqrt{s}}} \]
\[ = \frac{1}{\epsilon^2} \frac{-x}{4(1 - x^2)} H[0, x] \]

Here residua were taken at $z_6 = -n - 1, n = 0, 1, ..$, and $H[0, x] = \ln(x)$ and $x = \frac{\sqrt{-s + 4} - \sqrt{-s}}{\sqrt{-s + 4} + \sqrt{-s}}$. 
Summary

- We have introduced to the representation of $L$-loop $N$-point Feynman integrals of general type
- The determination of the $\epsilon$-poles is generally solved
- The remaining problem is the evaluation of the multi-dimensional, finite MB-Integrals
- This is unsolved in the general case, ... so you have something to do if you like to ...

Problem: Determine the small mass limit of B5l2m2 or of any other of the 2-loop boxes for Bhabha scattering.
Prof. Gluza may check your solution.
He leaves soon.