AMBRE – a Mathematica package for the automatic derivation of Mellin-Barnes Representations for Feynman Integrals

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- **Introduction: Feynman integrals: M-point functions with L loops**
- **AMBRE and MB**
- **Examples**
- **Summary**

Loop momentum integrations with Feynman parameters for L-loop n-point functions

Consider an arbitrary L-loop integral $G(X)$ with loop momenta k_l, with E external legs with momenta p_e, and with N internal lines with masses m_i and propagators $1/D_i$,

$$G(X) = \frac{1}{(i\pi^{d/2})^L} \int \frac{d^D k_1 \ldots d^D k_L \quad X(k_1, \ldots, k_L)}{D_1^{\nu_1} \ldots D_i^{\nu_i} \ldots D_N^{\nu_N}}.$$

$$D_i = q_i^2 - m_i^2 = [\sum_{l=1}^{L} c_i^l k_l + \sum_{e=1}^{E} d_i^e p_e]^2 - m_i^2$$

The numerator may contain a tensor structure

$$X = (k_1 p_{e_1}) \cdots (k_L p_{e_L}) = (k_1^{\alpha_1} \cdots k_L^{\beta_L}) \; (p_{e_1}^{\alpha_1} \cdots p_{e_L}^{\beta_L})$$

A nice box with numerator, B5l3m($p_e.k_1$)

We used it for the determination if the small mass expansion.

$$\mathtt{B5l3m}(\mathtt{p_e}\cdot\mathtt{k_1}) = \frac{m^{4\epsilon}(-1)^{a_{12345}}\ e^{2\epsilon\gamma_E}}{\prod_{j=1}^{5}\Gamma[a_i]\Gamma[5-2\epsilon-a_{123}](2\pi i)^4}\int_{-i\infty}^{+i\infty}d\alpha\int_{-i\infty}^{+i\infty}d\beta\int_{-i\infty}^{+i\infty}d\gamma\int_{-i\infty}^{+i\infty}d\delta$$

$$(-s)^{(4-2\ \epsilon)-a_{12345}-\alpha-\beta-\delta}\ (-t)^{\delta}$$

$$\frac{\Gamma[-4+2\ \epsilon+a_{12345}+\alpha+\beta+\delta]}{\Gamma[6-3\ \epsilon-a_{12345}-\alpha]}\frac{\Gamma[-\alpha]\ \Gamma[-\beta]}{\Gamma[7-3\ \epsilon-a_{12345}-\alpha]\ \Gamma[5-2\ \epsilon-a_{123}]}\frac{\Gamma[-\delta]}{\Gamma[4-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\ \Gamma[5-2\ \epsilon-a_{1123}-2}$$

$$\frac{\Gamma[2-\epsilon-a_{13}-\alpha-\gamma]}{\Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]}\frac{\Gamma[4-2\ \epsilon-a_{12345}-\alpha-\beta-\delta-\gamma]}{\Gamma[9-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]}\Big\{(p_e\cdot p_3)\ \Gamma[1+a_4+\delta]\ \Gamma[6-3\ \epsilon-$$

$$\Gamma[4-2\ \epsilon-a_{1234}-\alpha-\beta-\delta]\ \Gamma[3-\epsilon-a_{12}-\alpha]\ \Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \delta-\gamma]\Gamma[9-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-$$

$$\Gamma[5-2\ \epsilon-a_{1123}-\gamma]\ \Gamma[4-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\ \Gamma[a_1+\gamma]\ \Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+\Gamma[a_4+\delta]\Big[-(p_e\cdot p_1)\ \Gamma[7-3\ \epsilon-$$

$$\Gamma[4-2\ \epsilon-a_{1234}-\alpha-\beta-\delta]\Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \delta-\gamma]\Gamma[9-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]$$

$$\Big[\Gamma[3-\epsilon-a_{12}-\alpha]\ \Gamma[5-2\ \epsilon-a_{1123}-\gamma]\Gamma[4-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\ \Gamma[a_1+\gamma]+\Gamma[2-\epsilon-a_{12}-\alpha]\ \Gamma[4-2\ \epsilon-a_{1123}-\gamma]$$

$$\Gamma[5-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\ \Gamma[1+a_1+\gamma]\Big]\ \Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+\Gamma[6-3\ \epsilon-a_{12345}-\alpha]\ \Gamma[3-\epsilon-a_{12}+\alpha]$$

$$\Gamma[5-2\ \epsilon-a_{1123}-\gamma]\ \Gamma[4-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\Gamma[a_1+\gamma]\Big[((p_e\cdot(p_1+p_2))\ \Gamma[5-2\ \epsilon-a_{1234}-\alpha-\beta-\delta]\Gamma[9-4\ \epsilon-$$

$$\Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]\Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+(p_e\cdot p_1)\ \Gamma[4-2\ \epsilon-a_{1234}-\alpha-\beta-\delta]$$

$$\Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \delta-\gamma]\Gamma[9-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]\Gamma[-1+\epsilon+a_{123}+\alpha+\delta+\gamma]\Big]\Big\}$$

The MB-representation has to be calculated explicitely at fixed indices, e.g.

$$B5l3md2 = \frac{B_2}{\epsilon^2} + \frac{B_1}{\epsilon} + B_0$$

General Tasks, first two steps automated by `MB.m`:

- Find a region of definiteness of the n-fold MB-integral

$$\Re(z_1) = -1/80, \Re(z_3) = -33/40, \Re(z_5) = -21/20, \Re(z_6) = -59/160, \Re(\epsilon) = -1/10!$$

- Then go to the physical region where $\epsilon << 1$ by distorting the integration path step by step (adding each crossed residuum – per residue this means one integral less!!!)
- Take integrals by sums over residua, i.e. introduce infinite sums
- Sum these infinite multiple series into some known functions of a given class, e.g. Nielsen polylogs, Harmonic polylogs or whatever is appropriate.

Algorithmic solution for isolating the singularities in $1/\epsilon$ follows Tausk, PLB.
The automatization of that: MB.m (M. Czakon)

$$\begin{aligned} B5l3md2 &\rightarrow MB(\text{4-dim,fin}) + MB_3(\text{3-dim,fin}) \\ &+ MB_{36}(\text{2-dim}, \epsilon^{-1}, \text{fin}) + MB_{365}(\text{1-dim}, \epsilon^{-2}, \epsilon^{-1}, \text{fin}) \\ &+ MB_5(\text{3-dim,fin}) \end{aligned}$$

After these preparations e.g.:

$$\begin{aligned} MB_{365}(\text{1-dim}, \epsilon^{-2}) &\sim \frac{1}{\epsilon^2}\frac{1}{2\pi i}\int dz_6 \frac{(-s)^{(-z_6-1)}\Gamma(-z_6)^3\Gamma(1+z_6)}{8\Gamma(-2z_6)} \\ &= \frac{1}{\epsilon^2}\sum_{n=0,\infty} -\frac{(-1)^n(-s)^n\Gamma(1+n)^3)}{8n!\Gamma(-2(-1-n))} \\ &= -\frac{1}{\epsilon^2}\frac{ArcSin(\sqrt{s}/2)}{2\sqrt{4-s}\sqrt{s}} \\ &= \frac{1}{\epsilon^2}\frac{-x}{4(1-x^2)}H[0,x] \end{aligned}$$

Here residua were taken at $z_6 = -n-1, n = 0, 1, ..,$ and $H[0,x] = \ln(x)$ and $x = \frac{\sqrt{-s+4}-\sqrt{-s}}{\sqrt{-s+4}+\sqrt{-s}}$.

Introduce Feynman parameters

$$\frac{1}{D_1^{\nu_1} D_2^{\nu_2} \ldots D_N^{\nu_N}} = \frac{\Gamma(\nu_1 + \ldots + \nu_N)}{\Gamma(\nu_1) \ldots \Gamma(\nu_N)} \int_0^1 dx_1 \ldots \int_0^1 dx_N \frac{x_1^{\nu_1 - 1} \ldots x_N^{\nu_N - 1} \delta(1 - x_1 \ldots - x_N)}{(x_1 D_1 + \ldots + x_N D_N)^{N_\nu}},$$

with $N_\nu = \nu_1 + \ldots \nu_N$.

The denominator of G contains, after introduction of Feynman parameters x_i, the momentum dependent function m^2 with index-exponent N_ν:

$$(m^2)^{-(\nu_1 + \ldots + \nu_N)} = (x_1 D_1 + \ldots + x_N D_N)^{-N_\nu} = (k_i M_{ij} k_j - 2 Q_j k_j + J)^{-N_\nu}$$

Here M is an (LxL)-matrix, $Q = Q(x_i, p_e)$ an L-vector and $J = J(x_i x_j, m_i^2, p_{e_j} p_{e_l})$.
M, Q, J are linear in x_i. The momentum integration is now simple:
Shift the momenta k such that m^2 has no linear term in $\bar{k}$:

$$\begin{aligned} k &= \bar{k} + (M^{-1}) Q, \\ m^2 &= \bar{k} M \bar{k} - Q M^{-1} Q + J. \end{aligned}$$

Finally, one gets for **Scalar integrals:**

$$G(1) \quad = \quad (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{D}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^{N} dx_j \; x_j^{\nu_j-1} \delta\left(1-\sum_{i=1}^{N} x_i\right) \frac{U(x)^{N_\nu - D(L+1)/2}}{F(x)^{N_\nu - DL/2}}$$

with

$$U(x) \quad = \quad (\text{det } M) \; (\rightarrow 1 \text{ for } L=1)$$

$$F(x) \quad = \quad (\text{det } M) \; \mu^2 \quad = \quad -(\text{det } M) \; J + Q \; \tilde{M} \; Q \; (\rightarrow -J + Q^2 \text{ for } L=1)$$

Trick for one-loop functions:

$U = \text{det } M = 1 = \sum x_i$ and so U ‘disaapears’ and the construct $F_1(x)$ is bilinear in $x_i x_j$:

$$F_1(x) = -J(\sum x_i) + Q^2 = \sum A_{ij} x_i x_j.$$

The vector integral differs by some numerator $k_i p_e$ and thus there is a single shift in the integrand

$$k \to \bar{k} + U(x)^{-1}\tilde{M}Q$$

the $\int d^d\bar{k}\ \bar{k}/(\bar{k}^2+\mu^2) \to 0$, and no further changes:

$$G(k_{1\alpha}) = (-1)^{N_\nu}\frac{\Gamma\left(N_\nu - \frac{D}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)}\int_0^1 \prod_{j=1}^{N} dx_j\ x_j^{\nu_j-1}\delta\left(1-\sum_{i=1}^{N}x_i\right)\frac{U(x)^{N_\nu-D(L+1)/2-1}}{F(x)^{N_\nu-DL/2}}\left[\sum_l \tilde{M}_{1l}Q_l\right]_\alpha ,$$

Here also a tensor integral:

$$\begin{aligned} G(k_{1\alpha}k_{2\beta}) \quad &= \quad (-1)^{N_\nu}\frac{\Gamma\left(N_\nu - \frac{D}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)}\int_0^1 \prod_{j=1}^{N} dx_j\ x_j^{\nu_j-1}\delta\left(1-\sum_{i=1}^{N}x_i\right)\frac{U(x)^{N_\nu-2-D(L+1)/2}}{F(x)^{N_\nu-DL/2}} \\ &\times\sum_l\left[[\tilde{M}_{1l}Q_l]_\alpha[\tilde{M}_{2l}Q_l]_\beta - \frac{\Gamma\left(N_\nu-\frac{D}{2}L-1\right)}{\Gamma\left(N_\nu-\frac{D}{2}L\right)}\frac{g_{\alpha\beta}}{2}U(x)F(x)\frac{(V_{1l}^{-1})^{+}(V_{2l}^{-1})}{\alpha_l}\right]. \end{aligned}$$

The 1-loop case will be used in the following L times for a sequential treatment of an L-loop integral (remember $\sum x_j D_j = k^2 - 2Qk + J$ and $F(x) = Q^2 - J$):

$$G([1, kp_e]) \quad = \quad (-1)^{N_\nu}\frac{\Gamma\left(N_\nu-\frac{D}{2}\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)}\int_0^1 \prod_{j=1}^{N} dx_j\ x_j^{\nu_j-1}\delta\left(1-\sum_{i=1}^{N}x_i\right)\frac{[1, Qp_e]}{F(x)^{N_\nu-D/2}}$$

Examples for one-loop F-polynomials

One-loop vertex:

$$F(t,m^2) = m^2(x_1+x_2)^2 + [-t]x_1x_2$$

one-loop box:

$$F(s,t,m^2) = m^2(x_1+x_2)^2 + [-t]x_1x_2 + [-s]x_3x_4$$

one-loop pentagon:

$$F(s,t,t',v_1,v_2,m^2) = m^2(x_1+x_3+x_4)^2 + [-t]x_1x_3 + [-t']x_1x_4 + [-s]x_2x_5 + [-v_1]x_3x_5 + [-v_2]x_2x_4$$

2-loop: B7l4m2, sub-loop with 2 off-shell legs (diagram see next page):

$$\begin{aligned} F^{-(a_{4567}-d/2)} \quad = \quad & \Big\{[-t]x_4x_7 + [-s]x_5x_6 + m^2(x_5+x_6)^2 \\ & +(m^2-Q_1^2)x_7(x_4+2x_5+x_6) + (m^2-Q_2^2)x_7x_5\Big\}^{-(a_{4567}-d/2)} \end{aligned}$$

2-loop: B5l2m2, sub-loop with 2 off-shell legs (diagram see p.4):

$$F_{2lines}(k_1^2,m^2) = m^2(x_3)^2 + [-k_1^2+m^2]x_1x_3$$

What to be done now?

Perform the x-integrations

Find an as-general-as-possible general formula

Make it ready for algorithmic analytical and/or numerical evaluation

Integrating the Feynman parameters – get MB-Integrals

In 2-loops, consider two subsequent sub-loops (the first: off-shell 1-loop, second on-shell 1-loop) and get e.g. for B7l4m2, the planar 2nd type 2-box:

$$K_{\text{1-loop Box,off}} = \frac{(-1)^{a_{4567}}\Gamma(a_{4567}-d/2)}{\Gamma(a_4)\Gamma(a_5)\Gamma(a_6)\Gamma(a_7)} \int_0^\infty \Pi_{j=4}^7 dx_j x_j^{a_j-1} \frac{\delta(1-x_4-x_5-x_6-x_7)}{F^{a_{4567}-d/2}}$$

where $a_{4567} = a_4 + a_5 + a_6 + a_7$ and the function F is characteristic of the diagram; here for the off-shell 1-box (2nd type):

$$\begin{aligned} F^{-(a_{4567}-d/2)} = & \Big\{[-t]x_4x_7 + [-s]x_5x_6 + m^2(x_5+x_6)^2 \\ & +(m^2-Q_1^2)x_7(x_4+2x_5+x_6) + (m^2-Q_2^2)x_7x_5\Big\}^{-(a_{4567}-d/2)} \end{aligned}$$

We want to apply now:

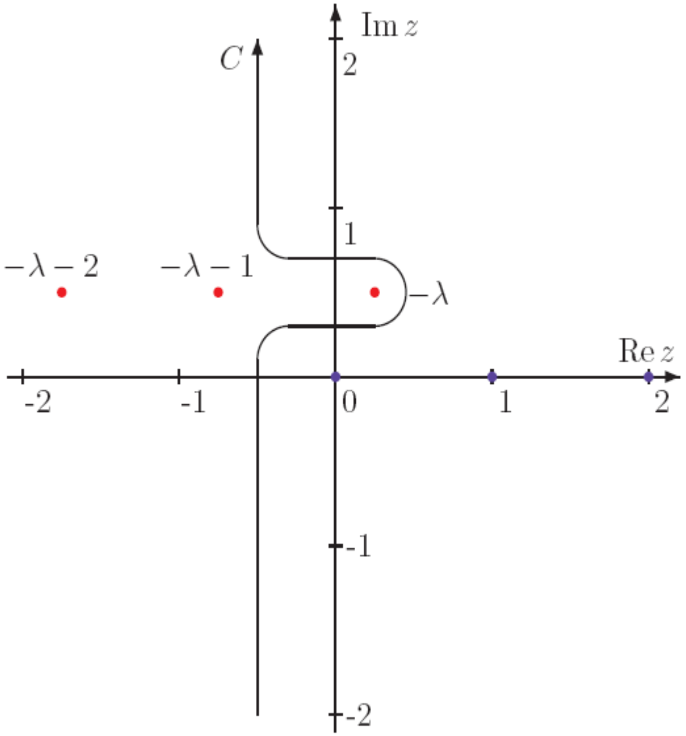
$$\int_0^1 \prod_{j=4}^7 dx_j \; x_j^{\alpha_j-1} \; \delta\left(1-x_4-x_5-x_6-x_7\right) = \frac{\Gamma(\alpha_4)\Gamma(\alpha_5)\Gamma(\alpha_6)\Gamma(\alpha_7)}{\Gamma\left(\alpha_4+\alpha_5+\alpha_6++\alpha_7\right)}$$

with coefficients α_i dependent on a_i and on F

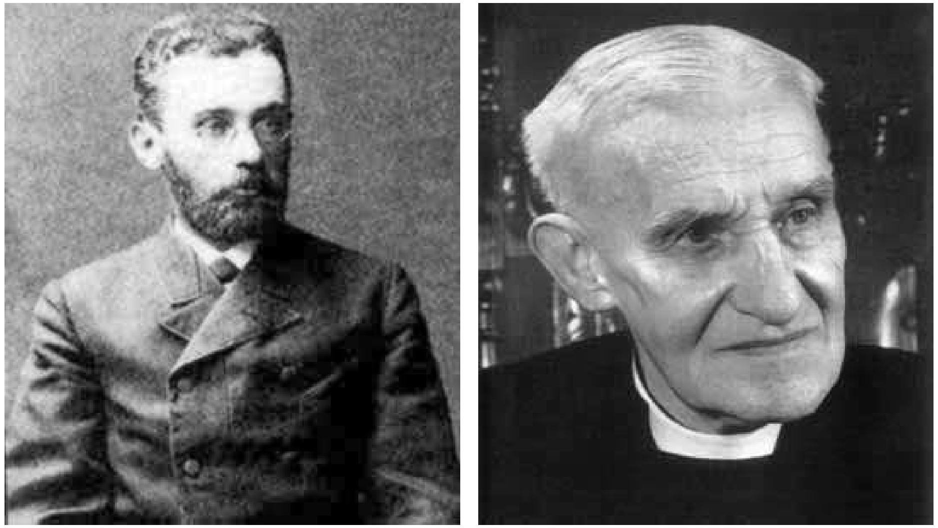
For this, we have to apply several MB-integrals here.

And do this, if needed, several times; here: repeat the procedure for the 2nd subloop.

$$\frac{1}{[A(s)x_1^{a_1}+B(s)x_1^{b_1}x_2^{b_2}]^a}=\frac{1}{2\pi i\;\Gamma(a)}\int\limits_{-i\infty}^{i\infty}d\sigma[A(s)x_1^{a_1}]^{\sigma}[B(s)x_1^{b_1}x_2^{b_2}]^{a+\sigma}\;\Gamma(a+\sigma)\Gamma(-\sigma)$$

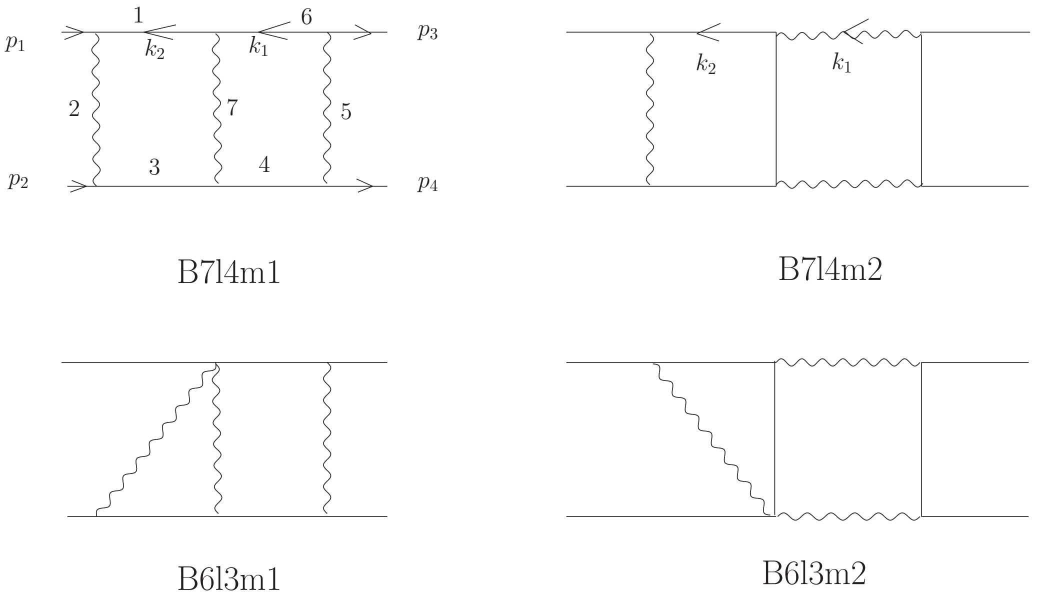


Mellin, Robert, Hjalmar, 1854-1933
Barnes, Ernest, William, 1874-1953



A short remark on history

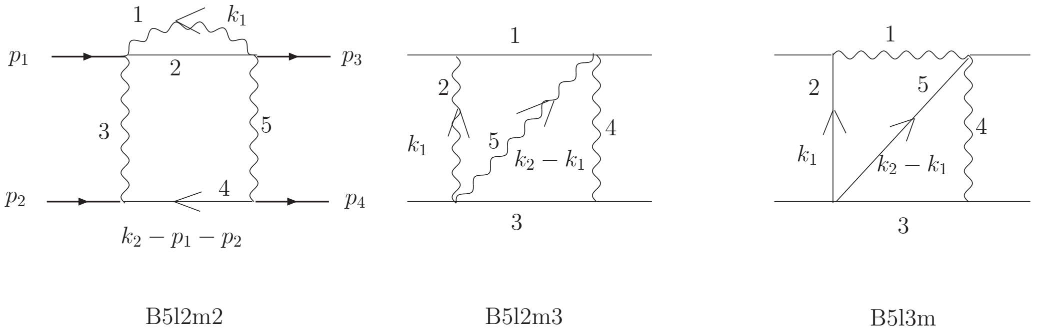
- N. Usyukina, 1975: "ON A REPRESENTATION FOR THREE POINT FUNCTION", Teor. Mat. Fiz. 22;
  a finite massless off-shell 3-point 1-loop function represented by 2-dimensional MB-integral
- E. Boos, A. Davydychev, 1990: "A Method of evaluating massive Feynman integrals", Theor. Math. Phys. 89 (1991);
  N-point 1-loop functions represented by n-dimensional MB-integral
- V. Smirnov, 1999: "Analytical result for dimensionally regularized massless on-shell double box", Phys. Lett. B460 (1999);
  treat UV and IR divergencies by analytical continuation: shifting contours and taking residues 'in an appropriate way'
- B. Tausk, 1999: "Non-planar massless two-loop Feynman diagrams with four on-shell legs", Phys. Lett. B469 (1999);
  nice algorithmic approach to that, starting from search for some unphysical space-time dimension d for which the MB-integral is finite and well-defined
- M. Czakon, 2005 (with experience from common work with J. Gluza and TR): "Automatized analytic continuation of Mellin-Barnes integrals", Comput. Phys. Commun. (2006);
  Tausk's approach realized in Mathematica program MB.m, published and available for use



B7l4m1 B7l4m2

B6l3m1 B6l3m2

Figure 1: The planar 6- and 7-line topologies.



B5l2m2 B5l2m3 B5l3m

Figure 2: The 5-line topologies. B7l4m2: shrink line 1 get B6l3m2, then line 4 get B5l3m

A nice box with numerator, B5l3m($p_e.k_1$)

We used it for the determination if the small mass expansion.

$$\texttt{B5l3m}(\texttt{p}_\texttt{e}\cdot\texttt{k}_1) = \frac{m^{4\epsilon}(-1)^{a_{12345}}\ e^{2\epsilon\gamma_E}}{\prod_{j=1}^{5}\Gamma[a_i]\Gamma[5-2\epsilon-a_{123}](2\pi i)^4}\int_{-i\infty}^{+i\infty}d\alpha\int_{-i\infty}^{+i\infty}d\beta\int_{-i\infty}^{+i\infty}d\gamma\int_{-i\infty}^{+i\infty}d\delta$$

$$(-s)^{(4-2\ \epsilon)-a_{12345}-\alpha-\beta-\delta}\ (-t)^{\delta}$$

$$\frac{\Gamma[-4+2\ \epsilon+a_{12345}+\alpha+\beta+\delta]}{\Gamma[6-3\ \epsilon-a_{12345}-\alpha]}\frac{\Gamma[-\alpha]\ \Gamma[-\beta]}{\Gamma[7-3\ \epsilon-a_{12345}-\alpha]\ \Gamma[5-2\ \epsilon-a_{123}]}\frac{\Gamma[-\delta]}{\Gamma[4-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\ \Gamma[5-2\ \epsilon-a_{1123}-2}$$

$$\frac{\Gamma[2-\epsilon-a_{13}-\alpha-\gamma]}{\Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]}\frac{\Gamma[4-2\ \epsilon-a_{12345}-\alpha-\beta-\delta-\gamma]}{\Gamma[9-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]}\Big\{(p_e\cdot p_3)\ \Gamma[1+a_4+\delta]\ \Gamma[6-3\ \epsilon-$$

$$\Gamma[4-2\ \epsilon-a_{1234}-\alpha-\beta-\delta]\ \Gamma[3-\epsilon-a_{12}-\alpha]\ \Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \delta-\gamma]\Gamma[9-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-$$

$$\Gamma[5-2\ \epsilon-a_{1123}-\gamma]\ \Gamma[4-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\ \Gamma[a_1+\gamma]\ \Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+\Gamma[a_4+\delta]\Big[-(p_e\cdot p_1)\ \Gamma[7-3\ \epsilon-$$

$$\Gamma[4-2\ \epsilon-a_{1234}-\alpha-\beta-\delta]\Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \delta-\gamma]\Gamma[9-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]$$

$$\Big[\Gamma[3-\epsilon-a_{12}-\alpha]\ \Gamma[5-2\ \epsilon-a_{1123}-\gamma]\Gamma[4-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\ \Gamma[a_1+\gamma]+\Gamma[2-\epsilon-a_{12}-\alpha]\ \Gamma[4-2\ \epsilon-a_{1123}-\gamma]$$

$$\Gamma[5-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\ \Gamma[1+a_1+\gamma]\Big]\ \Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+\Gamma[6-3\ \epsilon-a_{12345}-\alpha]\ \Gamma[3-\epsilon-a_{12}+\alpha]$$

$$\Gamma[5-2\ \epsilon-a_{1123}-\gamma]\ \Gamma[4-2\ \epsilon-a_{1123}-2\ \alpha-\gamma]\Gamma[a_1+\gamma]\Big[((p_e\cdot(p_1+p_2))\ \Gamma[5-2\ \epsilon-a_{1234}-\alpha-\beta-\delta]\Gamma[9-4\ \epsilon-a_{11223}$$

$$\Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]\Gamma[-2+\epsilon+a_{123}+\alpha+\delta+\gamma]+(p_e\cdot p_1)\ \Gamma[4-2\ \epsilon-a_{1234}-\alpha-\beta-\delta]$$

$$\Gamma[8-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \delta-\gamma]\Gamma[9-4\ \epsilon-a_{112233445}-2\ \alpha-2\ \beta-2\ \delta-\gamma]\Gamma[-1+\epsilon+a_{123}+\alpha+\delta+\gamma]\Big]\Big\}$$

Some routines in mathematica which were used:

```
(* Barnes' first lemma: \int d(si) Gamma(si1p+si)Gamma(si2p+si)Gamma(si1m-si)Gamma(si2m-si)
                        with 1/inv2piI = 2 Pi I *)

barne1[si_, si1p_, si2p_, si1m_, si2m_] :=
  1/inv2piI  Gamma[si1p + si1m] Gamma[si1p + si2m] Gamma[
      si2p + si1m] Gamma[si2p + si2m] /Gamma[si1p + si2p + si1m + si2m]

 (* Mellin-Barnes integral: (A+B)^(-nu) = 1/(2 Pi I) \int d(si) a^si b^(-nu - si)
                            Gamma[-si]Gamma[nu+si]/Gamma[nu]   *)

mb[a_, b_, nu_, si_]:=inv2piI a^si b^(-nu-si)Gamma[-si]Gamma[nu+si]/Gamma[nu]

 (* After the k-integration, the integrand for \int\prod(dxi xi^(ai -  1))\delta(1-\sum xi)
              will be (L=1 loop) : xfactorn F^(-nu) Q(xi).pe with  nu = a1 + .. + an - d/2 *)

xfactor3[a1_, x1_, a2_, x2_, a3_, x3_] :=
  I Pi^(d/2) (-1)^(a1 + a2 + a3) x1^(a1 - 1) x2^(a2 - 1)x3^(a3 - 1)Gamma[
        a1 + a2 + a3 - d/2]/(Gamma[a1]Gamma[a2]Gamma[a3])

 (* xinti - the i-dimensional x - integration over Feynman parameters /16 06 2005 *)

xint3[x1_^(a1_) x2_^(a2_) x3_^(a3_)  ] :=
  Gamma[a1 + 1] Gamma[a2 + 1] Gamma[a3 + 1]  / Gamma[a1 + a2 + a3 + 3]
```

DESY 07-037
HEPTOOLS 07-009
SFB/CPP-07-14

`AMBRE` – a Mathematica package for the construction of Mellin-Barnes representations for Feynman integrals

J. Gluza, K. Kajda, T. Riemann

AMBRE v.1.0 Abstract

The Mathematica toolkit `AMBRE` derives Mellin-Barnes (MB) representations for Feynman integrals in $d = 4 - 2\varepsilon$ dimensions. It may be applied for tadpoles as well as for multi-leg multi-loop scalar and tensor integrals. `AMBRE` uses a loop-by-loop approach and aims at lowest dimensions of the final MB representations. The present version of `AMBRE` works fine for planar Feynman diagrams. The output may be further processed by the package `MB` for the determination of its singularity structure in ε. The `AMBRE` package contains various sample applications for Feynman integrals with up to six external particles and up to four loops.

A `AMBRE` functions list

The basic functions of `AMBRE` are:

- **Fullintegral**[**{numerator},{propagators},{internal momenta}**] – is the basic function for input Feynman integrals
- **invariants** – is a list of invariants, e.g. **invariants = {p1*p1 → s}**
- **IntPart**[**iteration**] – prepares a subintegral for a given internal momentum by collecting the related numerator, propagators, integration momentum
- **Subloop**[**integral**] – determines for the selected subintegral the U and F polynomials and an MB-representation
- **ARint**[**result,i_**] – displays the MB-representation number i for Feynman integrals with numerators
- **Fauto**[**0**] – allows user specified modifications of the F polynomial **fupc**
- **BarnesLemma**[**repr,1,Shifts->True**] – function tries to apply Barnes' first lemma to a given MB-representation; when **Shifts->True** is set, `AMBRE` will try a simplifying shift of variables
  BarnesLemma[**repr,2,Shifts->True**] – function tries to apply Barnes' second lemma

.

AMBRE – **Automatic Mellin-Barnes Representations** for Feynman diagrams

For the Mathematica package AMBRE, many examples, and the program description, see:

http://prac.us.edu.pl/∼gluza/ambre/
http://www-zeuthen.desy.de/theory/research/CAS.html

Authors: J. Gluza, K. Kajda, T. Riemann

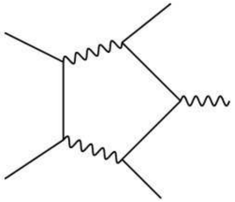
See also here:
http://www-zeuthen.desy.de/∼riemann/Talks/capp07/

with additional material presented at the CAPP – School on Computer Algebra in Particle Physics, DESY, Zeuthen, March 2007

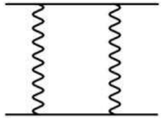
AMBRE - Automatic Mellin-Barnes REpresentation (arXiv:0704.2423)

To download 'right click' and 'save target as'.

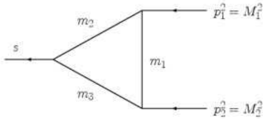
- The package AMBRE.m
- Kinematics generator for 4- 5- and 6- point functions with any external legs KinematicsGen.m
- Tarball with examples given below examples.tar.gz
  - example1.nb, example2.nb - Massive QED pentagon diagram.



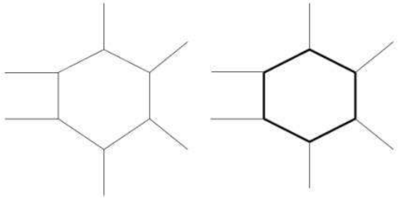
  - example3.nb - Massive QED one-loop box diagram.



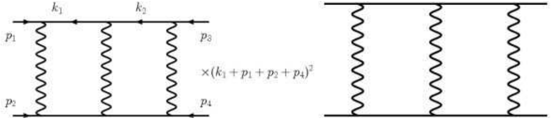
  - example4.nb - General one-loop vertex.



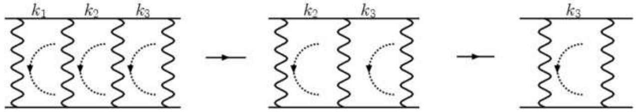
  - example5.nb - Six-point scalar functions;
    left: massless case,
    right: massive case.



  - example6.nb - **left**, example7.nb - **right**
    Massive two-loop planar QED box.



  - example8.nb - The loop-by-loop iterative procedure.



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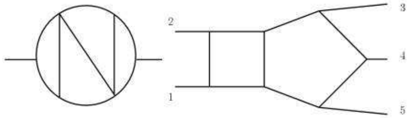
  ")
  !"



- $ ((&) $ (* & !"
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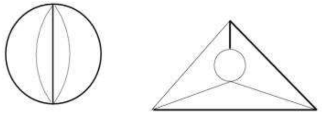

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V3l2m

The Feynman integral V3l2m is the QED one-loop vertex function, which is no master.
It is infrared-divergent (see this by counting of powers of loop integration momentum k or know it from: massless line between two external on-shell lines)

$$F = m^2(x_1+x_2)^2 + [-s]x_1x_2$$

Here: $s \equiv t$ (sorry!!!). We will also use the variable

$$y = \frac{\sqrt{-s+4}-\sqrt{-s}}{\sqrt{-s+4}+\sqrt{-s}}$$

$$\begin{aligned} \mathtt{V3l2m[y]} &= \frac{e^{\epsilon\gamma_E}\Gamma(-2\epsilon)}{2\pi i}\int dz(-s)^{-\epsilon-1-z}\frac{\Gamma^2(-\epsilon-z)\Gamma(-z)\Gamma(1+\epsilon+z)}{\Gamma(1-2\epsilon)\Gamma(-2\epsilon-2z)} \\ &= \frac{\mathtt{V3l2m[-1,y]}}{\epsilon} + \mathtt{V3l2m[0,y]} + \epsilon\ \mathtt{V3l2m[1,y]} + \cdots . \end{aligned} \tag{1}$$

$$
\begin{aligned}
\texttt{V3l2m}[-1,\texttt{y}] &= \frac{1}{2}\frac{1}{2\pi i}\int_{-i\infty+u}^{+i\infty+u} dr(-t)^{-1-r}\ \frac{\Gamma^3[-r]\Gamma[1+r]}{\Gamma[-2r)]} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(t)^n}{\begin{pmatrix}2n\\ n\end{pmatrix}(2n+1)} \\
&= \frac{1}{2}\frac{4\arcsin(\sqrt{t/2})}{\sqrt{4-t}\sqrt{t}} \\
&= \frac{1}{2}\frac{-2y(t)}{1-y^2(t)}\ln y(t) \qquad (2)
\end{aligned}
$$

Close path upwards to the left, so the infinite series of residua of

$$\Gamma[1+r]$$

at $r=-n, n=1,2,\cdots$ arises with weight function

$$G(r) = (-t)^{-1-r}\ \frac{\Gamma^3[-r]}{\Gamma[-2r]}$$

and the sum may be done with Mathematica, see p.33.

$$\begin{aligned}
\texttt{V3l2m[0,y]} &= \frac{1}{2\pi i}\int_{-i\infty+u}^{+i\infty+u} dr(-t)^{-1-r}\ \frac{\Gamma^3[-r]\Gamma[1+r]}{\Gamma[-2r]} \\
&\quad \frac{1}{2}\left[\gamma_E - \ln(-s) + 2\Psi[-2r] - 2\Psi[-r] + \Psi[1+r]\right] \qquad (3) \\
&= \frac{1}{2}\sum_{n=0}^{\infty} \frac{t^n}{\begin{pmatrix} 2n \\ n \end{pmatrix}(2n+1)} S_1(n),
\end{aligned}$$

and

$$
\begin{aligned}
\mathtt{V3l2m}[\mathtt{1},\mathtt{y}] \quad &= \quad \frac{1/4}{2\pi i}\int_{-i\infty+u}^{+i\infty+u} dr(-t)^{-1-r}\ \frac{\Gamma^3[-r]\Gamma[1+r]}{\Gamma[-2r)]} \\
&\quad \Big[\gamma_E^2 + Log[-s]^2 + Log[-s](-2\gamma_E - 4\Psi[-2z] + 4\Psi[-z] - 2\Psi[1+z]) \\
&\quad +\gamma_E(4\Psi[-2z] - 4\Psi[-z] + 2\Psi[1+z]) \\
&\quad -4\Psi[1,-2z] + 2\Psi[1,-z] + \Psi[1,1+z] \\
&\quad +4(\Psi[-2z]^2 - 2\Psi[-2z]\Psi[-z] + \Psi[-z]^2 + \Psi[-2z]\Psi[1+z] \\
&\quad -\Psi[-z]\Psi[1+z]) + \Psi[1+z]^2\Big] \\
&= \quad const\frac{1}{4}\sum_{n=0}^{\infty}\frac{(t)^n}{\begin{pmatrix}2n\\n\end{pmatrix}(2n+1)}\left[S_1(n)^2 + \zeta_2 - S_2(n)\right]. \qquad (4)
\end{aligned}
$$

Here, $\Psi[r] =$ ... and $\Psi[1,r] =$ ..., and the harmonic numbers $S_k(n)$ are

$$
S_k(n) \quad = \quad \sum_{i=1}^{n}\frac{1}{i^k},
$$

see e.g. talk by S.Moch.

Experimentally,

$$\mathtt{V312m}[2,\mathtt{y}] = \frac{1/12}{2\pi i}\int_{-i\infty+u}^{+i\infty+u} dz(-t)^{-1-z}\,\frac{\Gamma^3[-z]\Gamma[1+z]}{\Gamma[-2z)]}$$

$$\Big[a(z)+c_1(z)\Psi(0,1+z)+\Psi(2,1+z)+2\Psi(0,1+z)^2+\Psi(0,1+z)^3$$

$$+\,3\Psi(0,1+z)\Psi(1,1+z)+d_1(z)[\Psi(1,1+z)+2\Psi(0,1+z)^2]\Big] \qquad (5)$$

with some longer coefficients $cc1$, $d_1(z)$, aa:

```
c_1(z) =
3*EulerGamma^2 - 6*EulerGamma*Log[-s] + 3*Log[-s]^2
 + 12*PolyGamma[0, -2*z]^2
+  6*PolyGamma[0, -2*z]*(2*(EulerGamma - Log[-s]) - 4*PolyGamma[0, -z])
 - 12*(EulerGamma - Log[-s])*PolyGamma[0, -z]
 +  12*PolyGamma[0, -z]^2 - 12*PolyGamma[1, -2*z] + 6*PolyGamma[1, -z]
```

and

$$d_1(z) = 3*EulerGamma - 3*Log[-s] + 6*PolyGamma[0,-2*z] - 6*PolyGamma[0,-z] \qquad (6)$$

Finally,

```
a(z) =
EulerGamma^3 - 3*EulerGamma^2*Log[-s] + 3*EulerGamma*Log[-s]^2
- Log[-s]^3 + 8*PolyGamma[0, -2*z]^3
+  12*PolyGamma[0, -2*z]^2*(EulerGamma - Log[-s] - 2*PolyGamma[0, -z])
 + 12*(EulerGamma - Log[-s])*PolyGamma[0, -z]^2
 - 8*PolyGamma[0, -z]^3 - 12*EulerGamma*PolyGamma[1, -2*z]
 + 12*Log[-s]*PolyGamma[1, -2*z]
+  6*EulerGamma*PolyGamma[1, -z] - 6*Log[-s]*PolyGamma[1, -z]
 -  6*PolyGamma[0, -z]*(EulerGamma^2 - 2*EulerGamma*Log[-s] + Log[-s]^2
- 4*PolyGamma[1, -2*z] + 2*PolyGamma[1, -z])
+  6*PolyGamma[0, -2*z]*(EulerGamma^2 - 2*EulerGamma*Log[-s] + Log[-s]^2
 - 4*(EulerGamma - Log[-s])*PolyGamma[0, -z]
+    4*PolyGamma[0, -z]^2 - 4*PolyGamma[1, -2*z] + 2*PolyGamma[1, -z])
 + 8*PolyGamma[2, -2*z] - 2*PolyGamma[2, -z]
```

$$
\begin{aligned}
\mathtt{V3l2m}[2,\mathrm{y}] \quad = \quad & \frac{1}{1}\sum_{n=0}^{\infty}\frac{s^n}{\binom{2n}{n}(2n+1)} \\
& \Big[\frac{1}{12}S_1[n]^3-\frac{1}{4}S_1[n]S_2[n]+\frac{1}{4}\zeta_2 S_1[n]+\frac{1}{6}S_3[n]-\frac{1}{6}\zeta_3\Big]. \qquad (7)
\end{aligned}
$$

Sum this up!!

Answer is known to us from another technique: differential equations;
see our Bhabha webpage, file

master.m

Summary

- AMBRE: derive a representation of L-loop N-point planar Feynman integrals
- MB: The determination of the ϵ-poles is generally solved
- The remaining problem is the evaluation of the multi-dimensional, finite MB-Integrals
- This is unsolved in the general case