Massless and Massive Higher Loop Corrections

Johannes Blümlein

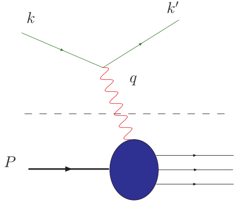
DESY

1. Introduction
2. Systematics in Loop Calculations
3. The differential 3-Loop World
4. Conclusions

Jos Fest, NIKHEF, Amsterdam, July 2015

Introduction

How precise has to be a calculation of an observable ?
This is often hard to answer, since experiments will improve significantly and some calculations can take quite long.



$\longrightarrow L_{\mu\nu}$ $\quad Q^2 := -q^2, \quad x := \frac{Q^2}{2P.q}$ Bjorken–x

$\longrightarrow W_{\mu\nu}$ $\quad \frac{d\sigma}{dQ^2\, dx} \sim W_{\mu\nu} L^{\mu\nu}$

DIS at HERA $\sim$ 1990 : only massless NLO corrections; Heavy quarks: LO. Will this be sufficient?

Present Goals :

- $\Delta\alpha_s(M_Z^2) \leq 1\%$ + very precise PDFs
- $\Longrightarrow$ of instrumental importance for $\sigma_{\rm Higgs}$ and m_t at the LHC.
- $\Longrightarrow$ of instrumental importance also for the measurement of the Higgs-bosons couplings to the matter and force fields.
- Theorists naturally would like to know the next order, to explore the QFTs further.

The new round :

1992 W. van Neerven & E.Zijlstra: massless 2-loop Wilson coefficients (–1994)
1991 S.Larin, F.Tkachov, J.Vermaseren The FORM-Version of 3-loop MINCER
1991/97 S.Larin & J.Vermaseren: The 3-loop DIS-sum rules
1994/2004 S.Larin, P.Nogueira, T. van Ritbergen, J.Vermaseren, A.Retey, JB:
The Moments of the 3-loop DIS anomalous dimensions and Wilson coefficients.

$$
\begin{aligned}
\gamma_{\rm NS}^{16,(2),+} &= -\left(\frac{58552930270652300886778705063429867}{345133797061245253431709667328000 0} - \frac{59290512768143}{1563722760600}\zeta_3\right) C_F^3 \\
&+\left(\frac{16704237280839842078788254 67}{648895948135156308787200 0} + \frac{59290512768143}{3127445521200}\zeta_3\right) C_F C_A^2 \\
&+\left(-\frac{12297946460007757811278560644 77}{303358855753185574358016 00000} - \frac{59290512768143}{1042481840400}\zeta_3\right) C_F^2 C_A \\
&+\left(-\frac{71543599677985155342551355451}{938967886855098206346240000} + \frac{64419601}{765765}\zeta_3\right) C_F^2 N_F \\
&+\left(-\frac{1501842182406038865943 6559}{579371382263532418560000} - \frac{64419601}{765765}\zeta_3\right) C_F C_A N_F \\
&-\frac{55594663498345731572 51}{20691835080840443520 00} C_F N_F^2 \\
&= 2849.5632736921273714 - 463.86001156801831223\, N_F - 3.5823897546153993659\, N_F^2\,.
\end{aligned}
$$

Introduction

The new round :

1992/1995 E.Laenen, W.van Neerven, S.Riemersma, J.Smith :
NLO Heavy Flavor Wilson coefficients
2000/01 W.van Neerven & A.Vogt : First numerical models of 3-loop anomalous dimensions
2001 - 2004 First NNLO QCD analyses of DIS data with quite different outcome.
The formula for α_s and other details are not unimportant here.

$\sim$ 2000:

Massless contributions: NNLO; Massive contributions: NLO.

Systematics in Loop Calculations: Function Spaces

1. The less-systematic era: (1965 - 1998)

$<$ 1970 Use of $\mathrm{Li}_n(f(x))$ L.Lewin 1958 & 1981: polylogarithms,

$$Li_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$$

1970 K.Kölbig, J.Mignaco, E.Remiddi : Nielsen integrals

$$\mathrm{S}_{n,p}(z) = \frac{(-1)^{p+n+1}}{(n-1)!p!} \int_0^1 \frac{dx}{x} \ln^{(n-1)}(x) \ln^p(1-xz)$$

$<$ 1998 Everybody used this.
Structures like [in 2-loop QCD]

$$\begin{aligned} F_1(x) &= \mathrm{S}_{1,2}\left(\frac{1-x}{2}\right) + \mathrm{S}_{1,2}(1-x) - \mathrm{S}_{1,2}\left(\frac{1-x}{1+x}\right) + \mathrm{S}_{1,2}\left(\frac{1}{1+x}\right) \\ &\quad - \ln(2)\mathrm{Li}_2\left(\frac{1-x}{2}\right) + \frac{1}{2}\ln^2(2)\ln\left(\frac{1+x}{2}\right) - \ln(2)\mathrm{Li}_2\left(\frac{1-x}{1+x}\right), \end{aligned}$$

are about the end, and one needs something more practical.

2 Loop Wilson Coefficients

Order α_s^2 contributions to the deep inelastic Wilson coefficient

W.L. van Neerven and E.B. Zijlstra
Instituut-Lorentz, University of Leyden, P.O. Box 9506, NL-2300 RA Leyden, The Netherlands

Volume 272, number 1,2 PHYSICS LETTERS B 28 November 1991

$$
\begin{aligned}
C_2^{(2),(+)}(x,1) = C_F^2\Big[&\frac{1+x^2}{1-x}\{4\ln^3(1-x)-(14\ln x+9)\ln^2(1-x)\\
&-[4\,\mathrm{Li}_2(1-x)-12\ln^2 x-12\ln x+16\zeta(2)+\tfrac{27}{2}]\ln(1-x)-\tfrac{4}{3}\ln^3 x-\tfrac{3}{2}\ln^2 x\\
&+[-24\,\mathrm{Li}_2(-x)+24\zeta(2)+\tfrac{61}{2}]\ln x+12\,\mathrm{Li}_3(1-x)-12S_{1,2}(1-x)\\
&+48\,\mathrm{Li}_3(-x)-6\,\mathrm{Li}_2(1-x)+32\zeta(3)+18\zeta(2)+\tfrac{51}{4}\}\\
&+(1+x)\{2\ln x\ln^2(1-x)+4[\mathrm{Li}_2(1-x)-\ln^2 x]\ln(1-x)\\
&-4[\mathrm{Li}_2(1-x)+\zeta(2)]\ln x+\tfrac{5}{3}\ln^3 x-4\,\mathrm{Li}_3(1-x)\}\\
&+\left(40+8x-48x^2-\tfrac{72}{5}x^3+\frac{8}{5x^2}\right)[\mathrm{Li}_2(-x)+\ln x\ln(1+x)]\\
&+(-8+40x)[\ln x\,\mathrm{Li}_2(-x)+S_{1,2}(1-x)-2\,\mathrm{Li}_3(-x)-\zeta(2)\ln(1-x)]+(5+9x)\ln^2(1-x)\\
&+\tfrac{1}{2}(-91+141x)\ln(1-x)-(28+44x)\ln x\ln(1-x)-(14+30x)\,\mathrm{Li}_2(1-x)\\
&+(\tfrac{29}{2}+\tfrac{25}{2}x+24x^2+\tfrac{36}{5}x^3)\ln^2 x+\frac{1}{10}\left(13-407x+144x^2-\frac{16}{x}\right)\ln x+(-10+6x-48x^2-\tfrac{72}{5}x^3)\zeta(2)\\
&+\tfrac{407}{20}-\tfrac{1917}{20}x+\tfrac{72}{5}x^2+\frac{8}{5x}+[6\zeta(2)^2-78\zeta(3)+69\zeta(2)+\tfrac{331}{8}]\delta(1-x)\Big]\\
+C_A C_F\Big[&\frac{1+x^2}{1-x}\{-\tfrac{11}{3}\ln^2(1-x)+[4\,\mathrm{Li}_2(1-x)+2\ln^2 x+\tfrac{44}{3}\ln x-4\zeta(2)+\tfrac{367}{18}]\ln(1-x)\\
&-\ln^3 x-\tfrac{55}{6}\ln^2 x+[4\,\mathrm{Li}_2(1-x)+12\,\mathrm{Li}_2(-x)-\tfrac{239}{6}]\ln x-12\,\mathrm{Li}_3(1-x)+12S_{1,2}(1-x)-24\,\mathrm{Li}_3(-x)\\
&+\tfrac{22}{3}\mathrm{Li}_2(1-x)+2\zeta(3)+\tfrac{22}{3}\zeta(2)-\tfrac{3155}{108}\}\\
&+4(1+x)[\mathrm{Li}_2(1-x)+\ln x\ln(1-x)]+\left(-20-4x+24x^2+\tfrac{36}{5}x^3-\frac{4}{5x^2}\right)[\mathrm{Li}_2(-x)+\ln x\ln(1+x)]\\
&+(4-20x)[\ln x\,\mathrm{Li}_2(-x)+S_{1,2}(1-x)-2\,\mathrm{Li}_3(-x)-\zeta(2)\ln(1-x)]+(\tfrac{133}{6}-\tfrac{1113}{18}x)\ln(1-x)\\
&+(-2+2x-12x^2-\tfrac{18}{5}x^3)\ln^2 x+\frac{1}{30}\left(13+1753x-216x^2+\frac{24}{x}\right)\ln x+(-2-10x+24x^2+\tfrac{36}{5}x^3)\zeta(2)\\
&-\tfrac{9687}{540}+\tfrac{59157}{540}x-\tfrac{36}{5}x^2-\frac{4}{5x}+[\tfrac{71}{5}\zeta(2)^2+\tfrac{140}{3}\zeta(3)-\tfrac{251}{3}\zeta(2)-\tfrac{5465}{72}]\delta(1-x)\Big]\\
+n_f C_F\Big(&\frac{1+x^2}{1-x}[\tfrac{2}{3}\ln^2(1-x)-(\tfrac{8}{3}\ln x+\tfrac{29}{9})\ln(1-x)-\tfrac{4}{3}\mathrm{Li}_2(1-x)+\tfrac{5}{3}\ln^2 x+\tfrac{19}{3}\ln x-\tfrac{4}{3}\zeta(2)+\tfrac{247}{54}]\\
&+\tfrac{1}{3}(1+13x)\ln(1-x)-\tfrac{1}{3}(7+19x)\ln x-\tfrac{23}{18}-\tfrac{27}{2}x+[\tfrac{4}{3}\zeta(3)+\tfrac{38}{3}\zeta(2)+\tfrac{457}{36}]\delta(1-x)\Big),
\end{aligned}
\qquad (9)
$$

where C_A, C_F denote the colour factors and n_f stands for the number of flavours. Here we have put $\mu^2=Q^2$. The more general case ($\mu^2\neq Q^2$) can be easily derived using renormalization group methods (see ref. [14]). In the above expression the terms of the type $\ln^i(1-x)/(1-x)$ have to be understood in the distributional sense [12]. The latter and the coefficient of the delta function can be derived from eq. (16) in ref. [13]. The second part in (8) is given by

$$
\begin{aligned}
C_2^{(2),G}(x,1) = n_f C_F\Big[&8(1+x)^2\\
&\times[-4S_{1,2}(-x)-4\ln(1+x)\,\mathrm{Li}_2(-x)-2\zeta(2)\ln(1+x)-2\ln x\ln^2(1+x)+\ln^2 x\ln(1+x)]\\
&+4(1-x)^2\{\tfrac{5}{6}\ln^3(1-x)-(2\ln x+\tfrac{13}{4})\ln^2(1-x)+[2\,\mathrm{Li}_2(1-x)+2\ln^2 x+4\ln x+\tfrac{7}{2}]\ln(1-x)-\tfrac{5}{12}\ln^3 x\\
&+[\mathrm{Li}_2(1-x)-4\,\mathrm{Li}_2(-x)+3\zeta(2)]\ln x-4\,\mathrm{Li}_3(1-x)-S_{1,2}(1-x)+12\,\mathrm{Li}_3(-x)+13\zeta(3)+\tfrac{13}{2}\zeta(2)\}\\
&+x^2\{\tfrac{10}{3}\ln^3(1-x)-12\ln x\ln^2(1-x)+[16\ln^2 x-16\zeta(2)]\ln(1-x)-5\ln^3 x\\
&+[12\,\mathrm{Li}_2(1-x)+20\zeta(2)]\ln x-8\,\mathrm{Li}_3(1-x)+12S_{1,2}(1-x)\}\\
&+\left(48+\tfrac{64}{3}x+\tfrac{96}{5}x^3+\frac{8}{15x^2}\right)[\mathrm{Li}_2(-x)+\ln x\ln(1+x)]+(14x-23x^2)\ln^2(1-x)\\
&+(-12x+10x^2)\ln(1-x)+(-24x+56x^2)\ln x\ln(1-x)+64x\,\mathrm{Li}_3(-x)+(-10+24x)\mathrm{Li}_2(1-x)\\
&+(-\tfrac{3}{2}+\tfrac{22}{3}x-36x^2-\tfrac{48}{5}x^3)\ln^2 x+\frac{1}{15}\left(-236+339x-648x^2-\frac{8}{x}\right)\ln x+(64x+36x^2)\zeta(3)\\
&+(-\tfrac{20}{3}x+46x^2+\tfrac{96}{5}x^3)\zeta(2)-\tfrac{647}{15}+\tfrac{239}{5}x-\tfrac{36}{5}x^2+\frac{8}{15x}\Big]\\
+n_f C_A\Big\{&4(1+x)^2[S_{1,2}(1-x)-2\,\mathrm{Li}_3(-x)+4S_{1,2}(-x)-2\ln x\,\mathrm{Li}_2(1-x)+4\ln(1+x)\,\mathrm{Li}_2(-x)\\
&+2\ln x\,\mathrm{Li}_2(-x)+2\zeta(2)\ln(1+x)+2\ln x\ln^2(1+x)+\ln^2 x\ln(1+x)]\\
&+8(1+2x+2x^2)\left[\mathrm{Li}_3\left(\frac{1-x}{1+x}\right)-\mathrm{Li}_3\left(-\frac{1-x}{1+x}\right)-\ln(1-x)\,\mathrm{Li}_2(-x)-\ln x\ln(1-x)\ln(1+x)\right]\\
&+\left(-24+\tfrac{80}{3}x^2-\frac{16}{3x}\right)[\mathrm{Li}_2(-x)+\ln x\ln(1+x)]+x^2[-4S_{1,2}(1-x)+16\,\mathrm{Li}_3(-x)+8\ln x\,\mathrm{Li}_2(1-x)\\
&+8\ln^2 x\ln(1+x)]+\tfrac{2}{3}(1-2x+2x^2)\ln^3(1-x)+(24x-8x^2)\ln x\ln^2(1-x)\\
&+\left(-2+36x-\tfrac{122}{3}x^2+\frac{8}{3x}\right)\ln^2(1-x)+(-4-32x+8x^2)\ln^2 x\ln(1-x)\\
&+(8-144x+148x^2)\ln x\ln(1-x)+(4+40x-8x^2)\ln(1-x)\,\mathrm{Li}_2(1-x)\\
&+(-20+24x-32x^2)\zeta(2)\ln(1-x)+\frac{1}{9}\left(-186-1362x+1570x^2+\frac{104}{x}\right)\ln(1-x)\\
&+(-4-72x+8x^2)\,\mathrm{Li}_3(1-x)+\frac{1}{3}\left(12-192x+176x^2+\frac{16}{x}\right)\mathrm{Li}_2(1-x)+\tfrac{1}{3}(10+28x)\ln^3 x\\
&+(-1+88x-\tfrac{194}{3}x^2)\ln^2 x+(-48x+16x^2)\zeta(2)\ln x+(58+\tfrac{584}{3}x-\tfrac{2090}{9}x^2)\ln x-(10+12x+12x^2)\zeta(3)\\
&+\frac{1}{3}\left(12-240x+268x^2-\frac{32}{x}\right)\zeta(2)+\tfrac{239}{9}+\tfrac{1072}{9}x-\tfrac{4493}{27}x^2+\frac{344}{27x}\Big\},
\end{aligned}
\qquad (5)
$$

W. van Neerven et al.: (1992) 79 functions

80 objects would be mathematically maximal.

2. The systematic era: ($>$ 1998)

2-loop Wilson coefficients in Mellin space are systematically expressed in terms of Harmonic Sums.

J.Vermaseren 1998 `summer.h'` paper 269 citations; JB & S.Kurth 1998

$$S_{b,\vec{a}}(N) = \sum_{k=1}^{N} \frac{(\mathrm{sign}(b))^k}{k^{|b|}} S_{\vec{a}}(k), \quad S_{\emptyset} = 1, \quad b, a_i \in \mathbb{Z}\backslash\{0\}$$

1999 E. Remiddi & J.Vermaseren: Harmonic Polylogartithms: `harmpol.h`

$$\int_0^1 dx x^N H_{\vec{a}}(x) \equiv \mathbf{M}[H_{\vec{a}}(x)](N) = S_{\vec{b}}(N) + ...$$

Iterated integrals : originally introduced by E.E.Kummer 1840 [as highschool teacher at Liegnitz, Silesia]; H.Poincaré 1884.

Both sets of objects form so-called (quasi)shuffle algebras and each obey as well more special relations.

Heavily used in many calculations:

e.g. : L.W. Garland, T. Gehrmann, E.W.N. Glover, A. Koukoutsakis and E. Remiddi, 2001:

$$
\begin{aligned}
&+27\mathrm{G}(0,1-z;y)-18\mathrm{G}(0,-z,1-z;y)\Big]+\frac{1}{9(y+z)}\Big[-\frac{3\pi^2}{2}\mathrm{H}(1;z)+\frac{3\pi^2}{2}\mathrm{G}(1-z;y)-170\\
&-18\mathrm{H}(0;z)\mathrm{G}(1-z;y)+9\mathrm{H}(0;z)\mathrm{G}(1-z,0;y)-72\mathrm{H}(0,1;z)+9\mathrm{H}(0,1,0;z)-123\mathrm{H}(1;z)\\
&-90\mathrm{H}(1;z)\mathrm{G}(-z;y)+18\mathrm{H}(1;z)\mathrm{G}(0;y)+18\mathrm{H}(1,0;z)+9\mathrm{H}(1,0;z)\mathrm{G}(1-z;y)-9\mathrm{H}(1,0;z)\mathrm{G}(0;y)\\
&-9\mathrm{H}(1,1,0;z)+123\mathrm{G}(1-z;y)-18\mathrm{G}(1-z,0;y)-9\mathrm{G}(1-z,1,0;y)+90\mathrm{G}(-z,1-z;y)\\
&-18\mathrm{G}(0,1-z;y)-9\mathrm{G}(0,1,0;y)\Big]+\frac{T\pi^2}{72}\Big[-115-24\mathrm{H}(0;z)\mathrm{G}(1-z;y)-12\mathrm{H}(0;z)\mathrm{G}(1;y)-12\mathrm{H}(0,1;z)\\
&+7\mathrm{H}(1;z)-48\mathrm{H}(1;z)\mathrm{G}(1-z;y)+36\mathrm{H}(1;z)\mathrm{G}(0;y)-12\mathrm{H}(1;z)\mathrm{G}(1;y)-12\mathrm{H}(1,0;z)+12\mathrm{H}(1,1;z)\\
&+48\mathrm{G}(1-z,1-z;y)+19\mathrm{G}(1-z;y)-24\mathrm{G}(1-z,0;y)-36\mathrm{G}(0,1-z;y)+48\mathrm{G}(0,1;y)\\
&+12\mathrm{G}(1,1-z;y)-26\mathrm{G}(1;y)+36\mathrm{G}(1,0;y)-48\mathrm{G}(1,1;y)\Big]+\frac{T}{54}\Big[-\frac{3\pi^4}{8}+\frac{15251}{12}-\frac{357}{2}\zeta_3\\
&+108\zeta_3\mathrm{H}(1;z)-270\zeta_3\mathrm{G}(1-z;y)+162\zeta_3\mathrm{G}(1;y)-360\mathrm{H}(0;z)-198\mathrm{H}(0;z)\mathrm{G}(1-z,1-z;y)\\
&+108\mathrm{H}(0;z)\mathrm{G}(1-z,1-z,0;y)+108\mathrm{H}(0;z)\mathrm{G}(1-z,-z,1-z;y)+78\mathrm{H}(0;z)\mathrm{G}(1-z;y)\\
&+54\mathrm{H}(0;z)\mathrm{G}(1-z,0,1-z;y)-180\mathrm{H}(0;z)\mathrm{G}(1-z,0;y)-108\mathrm{H}(0;z)\mathrm{G}(1-z,0,0;y)\\
&+54\mathrm{H}(0;z)\mathrm{G}(1-z,1,0;y)+108\mathrm{H}(0;z)\mathrm{G}(-z,1-z,1-z;y)+297\mathrm{H}(0;z)\mathrm{G}(-z,1-z;y)\\
&-108\mathrm{H}(0;z)\mathrm{G}(-z,-z,1-z;y)-108\mathrm{H}(0;z)\mathrm{G}(0,1-z,1-z;y)-180\mathrm{H}(0;z)\mathrm{G}(0,1-z;y)\\
&+216\mathrm{H}(0;z)\mathrm{G}(0,-z,1-z;y)-216\mathrm{H}(0;z)\mathrm{G}(0;y)+54\mathrm{H}(0;z)\mathrm{G}(0,1,0;y)-54\mathrm{H}(0;z)\mathrm{G}(1,1-z,0;y)\\
&-108\mathrm{H}(0;z)\mathrm{G}(1,0,1-z;y)+9\mathrm{H}(0;z)\mathrm{G}(1,0;y)+108\mathrm{H}(0;z)\mathrm{G}(1,0,0;y)-378\mathrm{H}(0,0;z)\mathrm{G}(1-z;y)\\
&-108\mathrm{H}(0,0;z)\mathrm{G}(1-z,0;y)-108\mathrm{H}(0,0;z)\mathrm{G}(0,1-z;y)+18\mathrm{H}(0,0,1;z)+216\mathrm{H}(0,0,1;z)\mathrm{G}(1-z;y)\\
&-432\mathrm{H}(0,0,1;z)\mathrm{G}(-z;y)-108\mathrm{H}(0,0,1;z)\mathrm{G}(0;y)+54\mathrm{H}(0,0,1;z)\mathrm{G}(1;y)+216\mathrm{H}(0,0,1,0;z)\\
&+348\mathrm{H}(0,1;z)+324\mathrm{H}(0,1;z)\mathrm{G}(1-z,-z;y)-279\mathrm{H}(0,1;z)\mathrm{G}(1-z;y)-162\mathrm{H}(0,1;z)\mathrm{G}(1-z,0;y)\\
&+108\mathrm{H}(0,1;z)\mathrm{G}(-z,1-z;y)-540\mathrm{H}(0,1;z)\mathrm{G}(-z,-z;y)-63\mathrm{H}(0,1;z)\mathrm{G}(-z;y)\\
&+108\mathrm{H}(0,1;z)\mathrm{G}(-z,0;y)+54\mathrm{H}(0,1;z)\mathrm{G}(0,1-z;y)-108\mathrm{H}(0,1;z)\mathrm{G}(0,-z;y)-117\mathrm{H}(0,1;z)\mathrm{G}(0;y)\\
&+108\mathrm{H}(0,1;z)\mathrm{G}(0,0;y)-54\mathrm{H}(0,1;z)\mathrm{G}(1,1-z;y)+54\mathrm{H}(0,1;z)\mathrm{G}(1,0;y)-252\mathrm{H}(0,1,0;z)\\
&-216\mathrm{H}(0,1,0;z)\mathrm{G}(1-z;y)+108\mathrm{H}(0,1,0;z)\mathrm{G}(-z;y)-54\mathrm{H}(0,1,0;z)\mathrm{G}(0;y)-54\mathrm{H}(0,1,0;z)\mathrm{G}(1;y)\\
&+198\mathrm{H}(0,1,1;z)-108\mathrm{H}(0,1,1;z)\mathrm{G}(-z;y)+108\mathrm{H}(0,1,1,0;z)+17\mathrm{H}(1;z)\\
&+432\mathrm{H}(1;z)\mathrm{G}(1-z,-z,-z;y)-477\mathrm{H}(1;z)\mathrm{G}(1-z,-z;y)-108\mathrm{H}(1;z)\mathrm{G}(1-z,-z,0;y)\\
&+297\mathrm{H}(1;z)\mathrm{G}(1-z;y)-108\mathrm{H}(1;z)\mathrm{G}(1-z,0,-z;y)+198\mathrm{H}(1;z)\mathrm{G}(1-z,0;y)\\
&+162\mathrm{H}(1;z)\mathrm{G}(1-z,1,0;y)+216\mathrm{H}(1;z)\mathrm{G}(-z,1-z,-z;y)-396\mathrm{H}(1;z)\mathrm{G}(-z,1-z;y)\\
&-108\mathrm{H}(1;z)\mathrm{G}(-z,1-z,0;y)+216\mathrm{H}(1;z)\mathrm{G}(-z,-z,1-z;y)-648\mathrm{H}(1;z)\mathrm{G}(-z,-z,-z;y)\\
&+234\mathrm{H}(1;z)\mathrm{G}(-z,-z;y)+108\mathrm{H}(1;z)\mathrm{G}(-z,-z,0;y)+426\mathrm{H}(1;z)\mathrm{G}(-z;y)\\
&-108\mathrm{H}(1;z)\mathrm{G}(-z,0,1-z;y)+108\mathrm{H}(1;z)\mathrm{G}(-z,0,-z;y)-297\mathrm{H}(1;z)\mathrm{G}(-z,0;y)\\
&-54\mathrm{H}(1;z)\mathrm{G}(0,1-z,-z;y)+198\mathrm{H}(1;z)\mathrm{G}(0,1-z;y)-108\mathrm{H}(1;z)\mathrm{G}(0,-z,1-z;y)\\
&+108\mathrm{H}(1;z)\mathrm{G}(0,-z,-z;y)-297\mathrm{H}(1;z)\mathrm{G}(0,-z;y)+216\mathrm{H}(1;z)\mathrm{G}(0,-z,0;y)-78\mathrm{H}(1;z)\mathrm{G}(0;y)\\
&+108\mathrm{H}(1;z)\mathrm{G}(0,0,-z;y)+378\mathrm{H}(1;z)\mathrm{G}(0,0;y)-54\mathrm{H}(1;z)\mathrm{G}(0,1,0;y)-54\mathrm{H}(1;z)\mathrm{G}(1,1-z,-z;y)\\
&-54\mathrm{H}(1;z)\mathrm{G}(1,0,-z;y)-81\mathrm{H}(1;z)\mathrm{G}(1,0;y)-108\mathrm{H}(1;z)\mathrm{G}(1,0,0;y)-81\mathrm{H}(1,0;z)
\end{aligned}
$$

18

$$
\begin{aligned}
&+216\mathrm{H}(1,0;z)\mathrm{G}(1-z,1-z;y)-108\mathrm{H}(1,0;z)\mathrm{G}(1-z,-z;y)+117\mathrm{H}(1,0;z)\mathrm{G}(1-z;y)\\
&-162\mathrm{H}(1,0;z)\mathrm{G}(1-z,0;y)-108\mathrm{H}(1,0;z)\mathrm{G}(-z,1-z;y)+108\mathrm{H}(1,0;z)\mathrm{G}(-z,-z;y)\\
&-297\mathrm{H}(1,0;z)\mathrm{G}(-z;y)-216\mathrm{H}(1,0;z)\mathrm{G}(0,-z;y)+171\mathrm{H}(1,0;z)\mathrm{G}(0;y)\\
&+108\mathrm{H}(1,0;z)\mathrm{G}(0,0;y)+54\mathrm{H}(1,0;z)\mathrm{G}(1,1-z;y)+54\mathrm{H}(1,0;z)\mathrm{G}(1,0;y)-108\mathrm{H}(1,0,0;z)\mathrm{G}(1-z;y)\\
&-54\mathrm{H}(1,0,0,1;z)+360\mathrm{H}(1,0,1;z)-162\mathrm{H}(1,0,1;z)\mathrm{G}(1-z;y)-108\mathrm{H}(1,0,1;z)\mathrm{G}(-z;y)\\
&+54\mathrm{H}(1,0,1;z)\mathrm{G}(0;y)+54\mathrm{H}(1,0,1;z)\mathrm{G}(1;y)+108\mathrm{H}(1,0,1,0;z)-297\mathrm{H}(1,1;z)\\
&-216\mathrm{H}(1,1;z)\mathrm{G}(-z,-z;y)+396\mathrm{H}(1,1;z)\mathrm{G}(-z;y)+108\mathrm{H}(1,1;z)\mathrm{G}(-z,0;y)+108\mathrm{H}(1,1;z)\mathrm{G}(0,-z;y)\\
&-198\mathrm{H}(1,1;z)\mathrm{G}(0;y)+81\mathrm{H}(1,1,0;z)-378\mathrm{H}(1,1,0;z)\mathrm{G}(1-z;y)+108\mathrm{H}(1,1,0;z)\mathrm{G}(-z;y)\\
&+108\mathrm{H}(1,1,0;z)\mathrm{G}(0;y)-54\mathrm{H}(1,1,0;z)\mathrm{G}(1;y)+108\mathrm{H}(1,1,0,1;z)+162\mathrm{H}(1,1,1,0;z)\\
&-297\mathrm{G}(1-z,1-z;y)-198\mathrm{G}(1-z,1-z,0;y)-216\mathrm{G}(1-z,1-z,1,0;y)+477\mathrm{G}(1-z,-z,1-z;y)\\
&+108\mathrm{G}(1-z,-z,1-z,0;y)-432\mathrm{G}(1-z,-z,-z,1-z;y)+108\mathrm{G}(1-z,-z,0,1-z;y)\\
&-17\mathrm{G}(1-z;y)-198\mathrm{G}(1-z,0,1-z;y)+108\mathrm{G}(1-z,0,-z,1-z;y)+78\mathrm{G}(1-z,0;y)\\
&-378\mathrm{G}(1-z,0,0;y)+108\mathrm{G}(1-z,0,1,0;y)-162\mathrm{G}(1-z,1,1-z,0;y)-162\mathrm{G}(1-z,1,0,1-z;y)\\
&+81\mathrm{G}(1-z,1,0;y)+108\mathrm{G}(1-z,1,0,0;y)+396\mathrm{G}(-z,1-z,1-z;y)+108\mathrm{G}(-z,1-z,1-z,0;y)\\
&-216\mathrm{G}(-z,1-z,-z,1-z;y)-426\mathrm{G}(-z,1-z;y)+108\mathrm{G}(-z,1-z,0,1-z;y)+297\mathrm{G}(-z,1-z,0;y)\\
&-216\mathrm{G}(-z,-z,1-z,1-z;y)-234\mathrm{G}(-z,-z,1-z;y)-108\mathrm{G}(-z,-z,1-z,0;y)\\
&+648\mathrm{G}(-z,-z,-z,1-z;y)-108\mathrm{G}(-z,-z,0,1-z;y)+108\mathrm{G}(-z,0,1-z,1-z;y)\\
&+297\mathrm{G}(-z,0,1-z;y)-108\mathrm{G}(-z,0,-z,1-z;y)-198\mathrm{G}(0,1-z,1-z;y)+54\mathrm{G}(0,1-z,-z,1-z;y)\\
&+78\mathrm{G}(0,1-z;y)-378\mathrm{G}(0,1-z,0;y)+162\mathrm{G}(0,1-z,1,0;y)+108\mathrm{G}(0,-z,1-z,1-z;y)\\
&+297\mathrm{G}(0,-z,1-z;y)-216\mathrm{G}(0,-z,1-z,0;y)-108\mathrm{G}(0,-z,-z,1-z;y)-216\mathrm{G}(0,-z,0,1-z;y)\\
&-360\mathrm{G}(0;y)-378\mathrm{G}(0,0,1-z;y)-108\mathrm{G}(0,0,-z,1-z;y)+108\mathrm{G}(0,0,1,0;y)+54\mathrm{G}(0,1,1-z,0;y)\\
&+54\mathrm{G}(0,1,0,1-z;y)+333\mathrm{G}(0,1,0;y)-216\mathrm{G}(0,1,1,0;y)+54\mathrm{G}(1,1-z,-z,1-z;y)\\
&+81\mathrm{G}(1,1-z,0;y)+108\mathrm{G}(1,1-z,0,0;y)+81\mathrm{G}(1,0,1-z;y)+108\mathrm{G}(1,0,1-z,0;y)\\
&+54\mathrm{G}(1,0,-z,1-z;y)+3\mathrm{G}(1,0;y)+108\mathrm{G}(1,0,0,1-z;y)+378\mathrm{G}(1,0,0;y)-216\mathrm{G}(1,0,1,0;y)\\
&+117\mathrm{G}(1,1,0;y)-216\mathrm{G}(1,1,0,0;y)+216\mathrm{G}(1,1,1,0;y)\Big]+\frac{\pi^2}{18}\Big[11+9\mathrm{H}(0;z)-24\mathrm{H}(0;z)\mathrm{G}(1-z;y)\\
&-6\mathrm{H}(0,1;z)+8\mathrm{H}(1;z)-36\mathrm{H}(1;z)\mathrm{G}(1-z;y)+24\mathrm{H}(1;z)\mathrm{G}(0;y)+6\mathrm{H}(1;z)\mathrm{G}(1;y)+24\mathrm{H}(1,1;z)\\
&+24\mathrm{G}(1-z,1-z;y)+2\mathrm{G}(1-z;y)-24\mathrm{G}(1-z,0;y)+12\mathrm{G}(1-z,1;y)-6\mathrm{G}(0,1-z;y)+9\mathrm{G}(0;y)\\
&+12\mathrm{G}(0,1;y)-6\mathrm{G}(1,1-z;y)-10\mathrm{G}(1;y)+24\mathrm{G}(1,0;y)-6\mathrm{G}(1,1;y)\Big]+\frac{1}{18}\Big[288\zeta_3+180\zeta_3\mathrm{H}(1;z)\\
&-360\zeta_3\mathrm{G}(1-z;y)+180\zeta_3\mathrm{G}(1;y)-188\mathrm{H}(0;z)-150\mathrm{H}(0;z)\mathrm{G}(1-z,1-z;y)\\
&+72\mathrm{H}(0;z)\mathrm{G}(1-z,1-z,0;y)+72\mathrm{H}(0;z)\mathrm{G}(1-z,-z,1-z;y)+295\mathrm{H}(0;z)\mathrm{G}(1-z;y)\\
&-300\mathrm{H}(0;z)\mathrm{G}(1-z,0;y)-36\mathrm{H}(0;z)\mathrm{G}(1-z,0,0;y)+72\mathrm{H}(0;z)\mathrm{G}(-z,1-z,1-z;y)\\
&+216\mathrm{H}(0;z)\mathrm{G}(-z,1-z;y)-72\mathrm{H}(0;z)\mathrm{G}(-z,-z,1-z;y)-36\mathrm{H}(0;z)\mathrm{G}(0,1-z,1-z;y)\\
&-156\mathrm{H}(0;z)\mathrm{G}(0,1-z;y)+36\mathrm{H}(0;z)\mathrm{G}(0,1-z,0;y)+144\mathrm{H}(0;z)\mathrm{G}(0,-z,1-z;y)+78\mathrm{H}(0;z)\mathrm{G}(0;y)\\
&-72\mathrm{H}(0;z)\mathrm{G}(0,0,1-z;y)+18\mathrm{H}(0;z)\mathrm{G}(0,0;y)+36\mathrm{H}(0;z)\mathrm{G}(0,1,0;y)-72\mathrm{H}(0;z)\mathrm{G}(1,1-z,0;y)
\end{aligned}
$$

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+ 30 other pages. harmpol.h: 466 citations.

Systematics in Loop Calculations: Function Spaces

These function spaces were fully sufficient for :

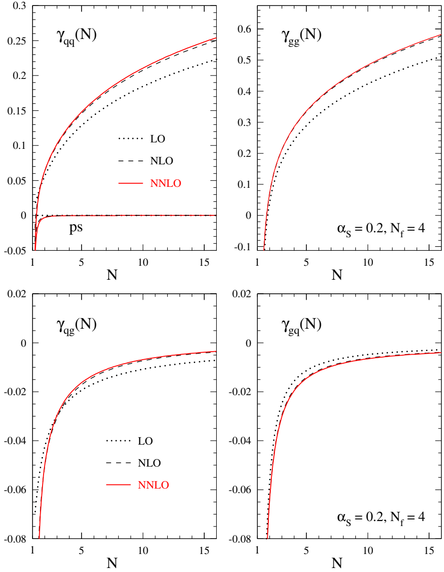
- all massless and single mass (asymptotic) 2-loop Wilson coefficients [$\leq$ 5 functions; earlier: 79 functions]
- the 3-loop anomalous dimensions [15 functions]
- the 3-loop massless Wilson coefficients [29 functions]

This is a great achievement.

The Differential 3-Loop World

- Individual moments are fine, but to know the whole function is better: Find the general N solution, i.e. one-dimensional distributions rather than just numbers.
- How does one get there ? $\Longrightarrow$ Difference equations. [See also below.]
- Here Jos performed pioneering work for the whole field, in an extremely smart manner.
- The method of the optical theorem allowed to determine both the 3-loop anomalous dimensions and the Wilson coefficients.
- 1st proof of evidence: repeat everything at the 2-loop level S.Moch & J.Vermaseren 1999.

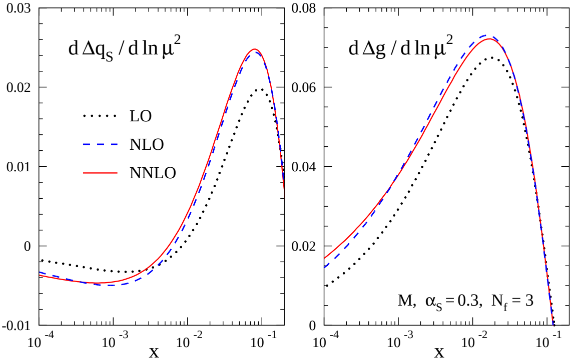
The 3-loop unpolarized anomalous dimensions



S.Moch, J.Vermaseren, and A.Vogt, 2004: 695 + 565 + 255 citations

The difference to the red line reduces $\Delta\alpha_s(M_Z^2)$ from 5 to 1%.

The 3-loop polarized anomalous dimensions



The improved evolution from LO to NNLO

(lhs of the Callan-Symanzik equations (1970) for PDF evolution).

S.Moch, J.Vermaseren, and A.Vogt, 2014

The previous results look simple, but they requested

- > 20 man years.
- Computer-algebra and computer power always at the very edge.
- various new algorithms.
- Harmonic sums up to weight $w = 6$ and HPLs up to weight $w = 5$.
- As a sign of grace the enormous amount of generalized harmonic sums just canceled in the end.
- Result: probably the largest formulae in QFT at the time (2005).

Application of these beautiful results:

DIS non-singlet World-data analysis

NLO	$\alpha_s(M_Z^2) = 0.1148 \pm 0.0020$
NNLO	$\alpha_s(M_Z^2) = 0.1134 \pm 0.0020$
N^3LO	$\alpha_s(M_Z^2) = 0.1141 \pm 0.0020 \pm 0.0007\ (\text{th})$

JB, H.Böttcher, A.Guffanti, 2006

From moments to expressions for general N :

If a physical quantity obeys a recursion, it always can be found based on a finite amount of moments. In case of no prejudice this is by Guessing, which has a rate of failure of $\sim 10^{-60}$.

Example: C_A^2 coefficient of $\gamma_{gg}^{(1)}$

Knowing 181 moments leads to a difference equation in N of the kind

$$\sum_{k=0}^{5}\left(\sum_{l=0}^{45} a_{k,l} N^l\right) f[N+k] = 0$$

(degree 45, order 5).

This equation can even be solved using Sigma [not always applying to possibly all occurring recurrences] and simplified by HarmonicSums to obtain

$$\gamma_{gg}^{(1),C_A^2} = 16S_{-2,1} - \frac{2P_2}{9(N-1)^2N^3(N+1)^3(N+2)^3} + \left[\frac{4P_1}{9(N-1)^2N^2(N+1)^2(N+2)^2} - 16S_2\right]S_1$$
$$+\frac{32(N^2+N+1)}{(N-1)N(N+1)(N+2)}S_2 + \left[\frac{32(N^2+N+1)}{(N-1)N(N+1)(N+2)} - 16S_1\right]S_{-2} - 8S_3 - 8S_{-3}$$

$$P_1 = 67N^8 + 268N^7 + 134N^6 - 392N^5 - 109N^4 + 844N^3 + 772N^2 - 144N - 144$$

$$P_2 = 48N^{11} + 336N^{10} + 1225N^9 + 144(-1)^N N^8 + 2886N^8 + 720(-1)^N N^7 + 4024N^7 + 1728(-1)^N N^6 + 2786N^6$$
$$+1800(-1)^N N^5 - 137N^5 - 1384N^4 - 1800(-1)^N N^3 + 552N^3 - 2016(-1)^N N^2 + 2576N^2 - 576(-1)^N N$$
$$+2064N + 576$$

We had physical cases to generate $\sim$ 1600 moments **and proceeded the above way.**

What about heavy flavor at NNLO ?

- $\Longrightarrow$ scaling violations differ a lot w.r.t. the massless case
- Thanks to a factorization derived by W. van Neerven and collab. (1995) at $Q^2/m_Q^2 \gtrsim 10$ for $F_2(x, Q^2)$, we may try to calculate these corrections.
- 2005 Asymptotic 3-Loop corrections for $F_L(x, Q^2)$ [JB, A.De Freitas, S.Klein, W. van Neerven]
- 2007 Systematic way to $F_2(x, Q^2)$ (2 Loops, general N) [I. Bierenbaum, JB, S.Klein]
- 2009 3-loop moments $N = 2, ..., 10(12, 14)$ were calculated. [I.Bierenbaum, JB, S.Klein]
- **Which structure have the 3-loop corrections for $F_2(x, Q^2)$ and how to get them ?**

An extesion of the package Reduze2 (A. von Manteuffel, C.Studerus, 2012) has been used to reduze the problem to master integrals.

They are computed using

- Generalized hypergeometric functions
- Modern Summation methods Sigma, EvaluateMultiSums, SumProduction (C.Schneider)
- Generating function techniques, various classes of special number and function space HarmonicSums (J.Ablinger)
- (multiple) Mellin-Barnes integrals
- The method of hyperlogarithms (F. Brown, 2008; F. Wißbrock, 2012)
- Systems of differential and difference equations
- Almqvist-Zeilberger theorem as integration method.

$F_2(x, Q^2)$

$$F_{(2,L)}^{\rm heavy}(x, N_F + 1, Q^2, m^2) =$$

$$x \sum_{k=1}^{N_F} e_k^2 \Biggl\{ L_{q,(2,L)}^{\rm NS}\left(x, N_F+1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) \otimes \left[f_k(x,\mu^2,N_F) + f_{\bar{k}}(x,\mu^2,N_F)\right]$$

$$+ \frac{1}{N_F} L_{q,(2,L)}^{\rm PS}\left(x, N_F+1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) \otimes \Sigma(x,\mu^2,N_F)$$

$$+ \frac{1}{N_F} L_{g,(2,L)}^{\rm S}\left(x, N_F+1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) \otimes G(x,\mu^2,N_F) \Biggr\}$$

$$e_Q^2 \Biggl[H_{q,(2,L)}^{\rm PS}\left(x, N_F+1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) \otimes \Sigma(x,\mu^2,N_F)$$

$$+ H_{g,(2,L)}^{\rm S}\left(x, N_F+1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) \otimes G(x,\mu^2,N_F) \Biggr] .$$

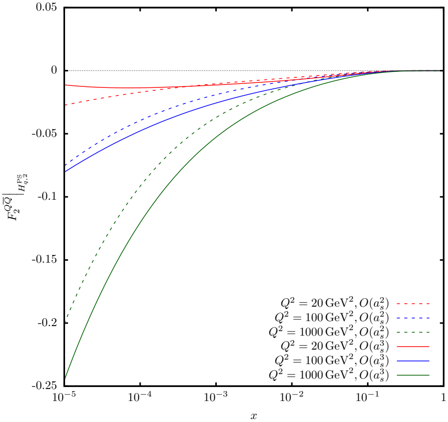
2010-2014

J. Ablinger, A. Behring, JB, A.De Freitas, A.Hasselhuhn, A.von Manteuffel, C.Raab, M.Round, C.Schneider, F. Wißbrock

DESY-RISC(Linz)-Mainz-IHES

$F_2(x, Q^2)$

The pure singlet contribution :



The small-x model of Ciafaloni, Catani, Hautmann (1991) does unfortunately not work (needed 23 years to be found).

Variable Flavor Number Scheme

$$f_k(n_f+1,\mu^2) + f_{\overline{k}}(n_f+1,\mu^2) = A_{qq,Q}^{\rm NS}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \left[f_k(n_f,\mu^2) + f_{\overline{k}}(n_f,\mu^2)\right]$$
$$+ \tilde{A}_{qq,Q}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f,\mu^2) + \tilde{A}_{qg,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes G(n_f,\mu^2)$$

$$f_{Q+\bar{Q}}(n_f+1,\mu^2) = \tilde{A}_{Qq}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f,\mu^2) + \tilde{A}_{Qg}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes G(n_f,\mu^2)\,.$$

$$G(n_f+1,\mu^2) = A_{gq,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f,\mu^2) + A_{gg,Q}^{S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes G(n_f,\mu^2)\,.$$

$$\Sigma(n_f+1,\mu^2) = \sum_{k=1}^{n_f+1}\left[f_k(n_f+1,\mu^2) + f_{\overline{k}}(n_f+1,\mu^2)\right]$$
$$= \left[A_{qq,Q}^{\rm NS}\left(n_f,\frac{\mu^2}{m^2}\right) + n_f \tilde{A}_{qq,Q}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right) + \tilde{A}_{Qq}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right)\right]$$
$$\otimes \Sigma(n_f,\mu^2)$$
$$+ \left[n_f \tilde{A}_{qg,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) + \tilde{A}_{Qg}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right)\right] \otimes G(n_f,\mu^2)$$

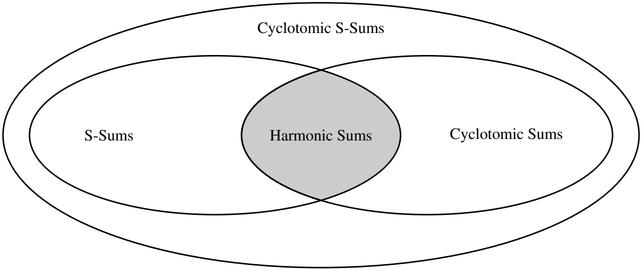
2010-2015 Important for PDF-descriptions at collider energies.

Emergence of new nested sums :

$$\sum_{i=1}^{N} \binom{2i}{i} (-2)^i \sum_{j=1}^{i} \frac{1}{j \binom{2j}{j}} S_{1,2}\left(\tfrac{1}{2}, -1; j\right) = \int_0^1 dx \frac{x^N - 1}{x - 1} \sqrt{\frac{x}{8+x}} \left[\mathrm{H}^*_{w_{17},-1,0}(x)\right.$$
$$\left. -2\mathrm{H}^*_{w_{18},-1,0}(x)\right] + \frac{\zeta_2}{2} \int_0^1 dx \frac{(-x)^N - 1}{x+1} \sqrt{\frac{x}{8+x}} \left[\mathrm{H}^*_{12}(x) - 2\mathrm{H}^*_{13}(x)\right]$$
$$+c_3 \int_0^1 dx \frac{(-8x)^N - 1}{x + \frac{1}{8}} \sqrt{\frac{x}{1-x}} \,,$$

... and (un)fortunately also non-iterative integrals ...

1



Nested (inverse) binomial sums ....

More and more onion skins to be added during theses calculations.

Spill-Off: New Mathematical Function Classes and Algebras

- 1998: Harmonic Sums [J.Vermaseren; JB]
- 1999: Harmonic Polylogarithms [E.Remiddi, J.Vermaseren]
- 1999: 2-dim. Harmonic Polylogarithms [T.Gehrmann and E.Remiddi]
- 2001: Generalized Harmonic Sums [S.Moch, P.Uwer, S.Weinzierl]
- 2004: Infinite harmonic (inverse) binomial sums [A.Davydychev, M.Kalmykov; S.Weinzierl]
- 2011: (generalized) Cyclotomic Harmonic Sums, polylogarithms and numbers [J.Ablinger, JB, C.Schneider]
- 2013: Systematic Theory of Generalized Harmonic Sums, polylogarithms and numbers [J.Ablinger, JB, C.Schneider]
- 2014: Finite nested Generalized Cyclotomic Harmonic Sums with (inverse) Binomial Weights [J.Ablinger, JB, C.Raab, C.Schneider]

Particle Physics Generates NEW Mathematics.

Jos at Loops and Legs

Rheinsberg 1998 Kloster Banz 2002

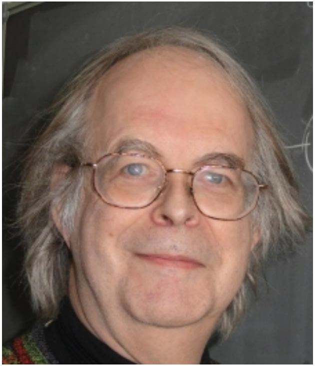
The β-summit : 1-4 loops
Zinnowitz 2004

Dear Jos,
we owe you an enormous amount of great things, foremost `Form` and `TForm`, and many other very useful packages.

We thank you for introducing several very useful function classes into QFT and making the biggest data mine of ζ-values [even being searched by people working on strings.]

We owe you and your collaborators $\beta^{(3)}_{\mathrm{SU(N)}}$ and all the nice 3-loop anomalous dimensions and Wilson coefficients, both unpolarized and polarized, constituting an epochal work.

... But as we all understand, there is even much more to come.



Happy Birthday, Jos, and many happy returns!