

# The $O(\alpha^2)$ initial state QED corrections to $e^+e^- \rightarrow \gamma^* Z^*$ at very high luminosity colliders

Johannes Blümlein  
DESY

[ In collaboration with A. De Freitas<sup>1</sup>, C.G. Raab<sup>2</sup>, K. Schönwald<sup>1,3</sup>]

<sup>1</sup>DESY,    <sup>2</sup>U. Linz,    <sup>3</sup> KIT

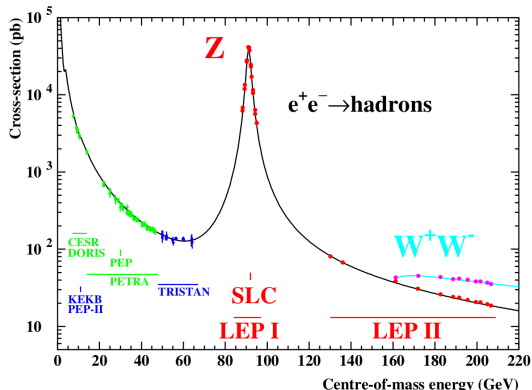
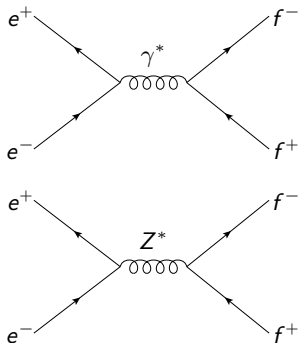
Dec. 3, 2020, NIKHEF TH Seminar, Amsterdam

based on:

J. Blümlein, A. De Freitas, C. Raab and K. Schönwald, Phys.Lett. B791 (2019) 206-209  
J. Blümlein, A. De Freitas, C. Raab and K. Schönwald, Phys.Lett. B801 (2020) 135196, Nucl.Phys.B 956 (2020) 115055, J. Blümlein, A. De Freitas, and K. Schönwald, Nucl.Phys.B 955 (2020) 115059 and in preparation.

- ▶ Introduction
- ▶ Initial State Radiation in  $e^+ e^-$  Annihilation
  - ▶ The History: 1987–2020
  - ▶ The Method of Massive Operator Matrix Elements
  - ▶ The Full Diagrammatic Calculation
- ▶ Phenomenological Results
- ▶ Conclusions and Outlook

# Introduction



- ▶ These corrections are important for the prediction of the Z-boson peak and for  $t\bar{t}$  production at LEP, ILC and FCC-ee, and at Higgs factories through  $e^+e^- \rightarrow Z^*H^0$ .
- ▶ We **revisit** the initial state corrections to  $e^+e^-$  annihilation to a (virtual) neutral vector boson, since only one comprehensive calculation existed: Berends, Burgers, van Neerven (Nucl. Phys. B297 (1988))

## Theory of Initial State Radiation

We look at the process:

$$e^- + e^+ \rightarrow \gamma^*/Z^* \rightarrow f^- + f^+$$

with the invariants

$$(p_- + p_+)^2 = s, \quad p_-^2 = p_+^2 = m_e^2, \quad (p_f + p_{\bar{f}})^2 = q^2 = s'$$

The initial state radiation (ISR) of  $n$  particles can be described by:

$$\frac{d\sigma}{ds'} = \frac{\sigma^0(s')}{4s} \mathcal{H} \left( \alpha, z \equiv \frac{s'}{s}, \rho \equiv \frac{m_e^2}{s} \right),$$

where  $\sigma^0(s')$  describes the leading order process  $e^+ e^- \rightarrow f \bar{f}$  and  $\mathcal{H}(\alpha, z, \rho)$  radiator functions described by the Drell-Yan process with **massive** initial states,  $\rho = m_e^2/s$ ,

$$\mathcal{H}(\alpha, z, \rho) = \delta(1-z) + \sum_{k=1}^{\infty} \left( \frac{\alpha}{4\pi} \right)^k \sum_{l=0}^k h_{kl} \ln^l(s/m_e^2)$$

# Why do we have to revisit these matters ?

## ISR corrections have been calculated up to $O(\alpha^2)$ in:

Nuclear Physics B207 (1982) 429-478  
North-Holland, Amsterdam

### HIGHER ORDER RADIATIVE CORRECTIONS AT LEP EXPERIMS

F.A. BERENDS and W.L. VAN SHERWYD

Amsterdam University, P.O. Box 5050, 1000 SB Amsterdam, The Netherlands

G.J.E. HORVATH

ORNL, Clinton, Tennessee

Received 12 August 1982

A complete leading  $O(\alpha^2)$  initial state radiative correction to the Z-resonance shape is presented. The correction is compared with those previously given only for the situation where one resonates in 13 orders of perturbation theory. Our results show that the radiative peak coordinate the half of the relative correction near the top of the Z-peak. The effect of non-physical QED processes on the Z-resonance is found to be very small. The cross sections have been obtained by means of a standard Feynman diagram calculation. In addition we have also computed the cross sections by using the renormalization group method, where besides the leading log  $N(\alpha^2)$  and the next-to-leading log are also taken into account.

### 1. Introduction

Electron-positron colliding beam experiments which will be carried out in the near future at SLC and LEP provide us with a wealth of information about the standard model of the electroweak interactions. A very accurate determination [1] exists about the subject one wants to investigate. For year months people search in the search for new particles like the Higgs boson, the top quark and maybe some superpartner partners of the particles in the standard model. Furthermore, one wants to make precise determinations of the electroweak parameters among which the most interesting are the mass and width of the Z boson. For the study of new physics effects, it is of the greatest importance to measure those quantities with a very high degree of accuracy. As has been extensively discussed in refs. [2, 3], the determination of the mass and width of the Z boson will be greatly affected by radiative corrections. In particular the pure QED part of the standard model leads to a distortion and a shift of the resonance peak. Up to now complete one-loop radiative corrections of the process  $e^+e^- \rightarrow e^+e^-$  in which the Z is produced have been carried out [4]. The calculations reveal that the bulk of the corrections can be attributed to photon contributions from the initial electron-positron state. Interpretation of the lowest order self photon correction shows [2, 3] that higher order

0551-0893/82/00184-0000/\$02.00 © North-Holland Publishing Company  
North-Holland Physics Publishing Division

Nuclear Phys. B207, number 1-2

PHYSICS LETTERS B

4 August 1982

### PERTURBATIVE QCD AND LEPTON PAIR PRODUCTION

### RADIATIVE CORRECTIONS TO $gg \rightarrow q\bar{q}$ RADIATIVE CORRECTIONS TO $e^+e^-$ ANNIHILATION

B.A. KATHEE, M. KRASCZYK<sup>1,2</sup>, J.H. KÜHN and R.G. STUART

1) The Ohio State University, Columbus, Ohio 43210, U.S.A.  
2) The Ohio State University, Columbus, Ohio 43210, U.S.A.

Received 11 April 1982

The complete hadronic corrections to  $O(\alpha^2)$  and part-structure corrections from initial state radiative to  $e^+e^-$  annihilation to standard model radiative corrections to the production and of hadronic systems initiated by  $O(\alpha^2)$  two-loop corrections to the  $gg \rightarrow q\bar{q}$  cross-sections are presented by the authors in this paper. A study is made of the influence of the corrections to the cross-sections in the region of the Z-peak and in the region of the continuum.

### 1. Introduction

Measurements of the total cross section for  $e^+e^-$  annihilation have reached a level of precision [1] where the influence of higher order radiative corrections is no longer negligible. Also the determination of the Z-resonance width through the resonance line shape at  $\sqrt{s} = m_Z$  will be influenced by radiative corrections and again the influence of  $O(\alpha^2)$  is non-negligible [2].

In those reaction radiative corrections are dominantly due to initial-state radiation. To  $O(\alpha^2)$  nearly planar corrections as well as those from soft and virtual photons and hadronic states are relevant. The logarithmic and soft-photon-like terms that are enhanced by powers of  $\ln(s/m_e^2)$  have been calculated in ref. [3] and the remaining constant terms, especially of order  $1/\epsilon^2$ , are given in ref. [4, 5].

Hadronic corrections are known to contribute approximately 50% to the large logarithms that appear in the next-to-leading  $O(\alpha^2)$  corrections [6]. They are therefore expected to be important also in the high-energy region for  $O(\alpha^2)$  radiative corrections in the aforementioned large logarithms. Just as for  $O(\alpha^2)$  those  $O(\alpha^2)$  hadronic corrections are also dominated by the quantity  $R(s) = \sigma_{had}(s)/\sigma_{pt}(s)$ , measured in lower energy  $e^+e^-$  collisions, since the  $R(s)$  approaches a constant value for large  $s$ .

By following the structure expansion for the hadronic contribution to the virtual  $O(\alpha^2)$  corrections which is valid for arbitrary  $\epsilon$ , it becomes particularly simple in the large- $\epsilon$  region. The information contained in  $R(s)$  can be summarized in an asymptotic behavior logarithmic in terms of  $\ln(s/m_e^2)$  and  $\ln(s/m_Z^2)$  terms. A similar approach will be developed for next soft and hard hadronic corrections which in the high energy region depends on the same moments. The formulas can be easily applied to the explicit case of lepton annihilation and separation nuclear nuclei for virtual radiation [3, 4] and the logarithmically enhanced terms from real radiation [2]. However, in the hard case, one depends on ref. [1] in radiative corrections.

A general relation arises also in the Z-peak. The three cross sections change rapidly when the energy varies by  $\pm 2 \text{ GeV}$  in QED. On the other hand, hadronic production of  $e^+e^-$  annihilations has been studied at low  $\sqrt{s}$ , and above 800 MeV and shows up-scale variations up to 10 GeV. The "soft" approximation that can be justified

is due to reference from Institute of Theoretical Physics, Warsaw University, 00-681, Warsaw, Poland.

Supported in part by Ministry of Science and Higher Education, Warsaw University, Poland (1981-82).

0551-0893/82/00184-0000/\$02.00 © North-Holland Publishing Company

North-Holland Physics Publishing Division

207

A. N. J. J. SCHELLEKENS

Berends et al.: Complete  $O(\alpha^2)$  ISR, (385 citations)  
Kniehl et al.: so-called  $O(\alpha^2)$  NS-corrections process II of BBN).

Are these widely used results correct ?

- ▶ Numerous calculations at  $O(\alpha)$  are in agreement, since about the time of PETRA and PEP.
- ▶ First higher order universal [leading log] results are calculated using QED-factorization.
- ▶ The accuracy reached at LEP requires the  $O(\alpha^2)$  corrections
- ▶ **1987**: First  $O(\alpha^2)$  calculation Berends, Burgers, van Neerven (Nucl. Phys. B297 (1988))
- ▶ **1990**:  $O(\alpha_s^2)$  QCD corrections to the massless Drell-Yan process  
Hamberg, van Neerven, Matsuura, (Nucl.Phys. B359 (1991))  
[Contains processes yet missing in Berends et al. (1987) and accounts for  $\gamma_5$  correctly]
- ▶ **~ 1996**: A first attempt to perform the  $O(\alpha^2)$  corrections using massive OMEs; Mertig, van Neerven, Scharf; led to tarcer. Calculation did not converge.



W.L. van Neerven: "I will get into all this, after they have gotten fermion-number conservation."

- ▶ **2002: A new start:** JB, De Freitas, van Neerven.
- ▶ At this time modern integration technologies for massive 2- and higher loop calculations, also in the inclusive case, did not yet exist. We just had harmonic sums and harmonic polylogarithms Vermaseren 1998, JB and Kurth 1998, Vermaseren and Remiddi 1999.  
We integrated in a rather **baroque and lengthy** way, applying a lot of tricks. In a way it was painful, but we wanted to get the result.
- ▶ **2007:** Willy worked on the project until one hour before he died and we still had exchanged e-mails. The work has been done shared at **Leiden, Caracas and Zeuthen**, e-mail and phone assisted.
- ▶ **2008:** We finished the calculation, but **disagreed with** Berends et al. in all the non logarithmic terms at  $O(\alpha^2)$ ; this also applied to Kniehl, Krawczyk, Kühn, Stuart, Phys. Lett. B209 (1988)  
We tried to clarify these differences for 3 years, without success.
- ▶ **2011:** JB, De Freitas, van Neerven, Nucl.Phys. B855 (2012)  
**Does the massive Drell-Yan process not factorize from 2-loop onward?**  
(There has been a proof of factorization by J. Collins.)
- ▶ **> June 2018:** After having a lot of experience in massive 2- and 3-Loop calculations in QCD, we decided to perform the QED  $O(\alpha^2)$  corrections **without any approximation to finally decide, which result is correct.**

## The Born Cross Section

The Born Cross Section :  $e^+e^- \rightarrow f, \bar{f}$   $f \neq e$

$$\frac{d\sigma^{(0)}(s)}{d\Omega} = \frac{\alpha^2}{4s} N_{C,f} \sqrt{1 - \frac{4m_f^2}{s}} \left[ \left( 1 + \cos^2 \theta + \frac{4m_f^2}{s} \sin^2 \theta \right) G_1(s) - \frac{8m_f^2}{s} G_2(s) + 2\sqrt{1 - \frac{4m_f^2}{s}} \cos \theta G_3(s) \right]$$

$$\sigma^{(0)}(s) = \frac{4\pi\alpha^2}{3s} N_{C,f} \sqrt{1 - \frac{4m_f^2}{s}} \left[ \left( 1 + \frac{2m_f^2}{s} \right) G_1(s) - 6\frac{m_f^2}{s} G_2(s) \right]$$

$$G_1(s) = Q_e^2 Q_f^2 + 2Q_e Q_f v_e v_f \operatorname{Re}[\chi_Z(s)] + (v_e^2 + a_e^2)(v_f^2 + a_f^2) |\chi_Z(s)|^2$$

$$G_2(s) = (v_e^2 + a_e^2) a_f^2 |\chi_Z(s)|^2$$

$$G_3(s) = 2Q_e Q_f a_e a_f \operatorname{Re}[\chi_Z(s)] + 4v_e v_f a_e a_f |\chi_Z(s)|^2.$$

$$\chi_Z(s) = \frac{s}{s - M_Z^2 + iM_Z\Gamma_Z}$$



## The $\mathcal{O}(\alpha)$ Corrections

- ▶ The first radiative corrections come from the process

$$e^+ + e^- \rightarrow \gamma^*/Z^* + \gamma.$$

- ▶ To stay in  $d = 4$ , we can split the contributions into hard, soft and virtual photons.
- ▶ The hard part is characterized by demanding  $k^0 > \frac{\sqrt{s}\Delta}{2}$ .
- ▶ The soft and virtual parts of the cross section have to be made infrared finite by introducing a small photon mass  $\lambda$ .
- ▶ The cross section is then given by

$$\frac{d\sigma^{(1),I}}{ds'} = \frac{d\sigma^{(0)}}{s} \left(\frac{\alpha}{\pi}\right) \left[ \delta(1-z) \left( \delta_1^{S1}(\lambda, \Delta) + \delta_1^{V1}(\lambda) \right) + \theta(1-z-\Delta) \delta_1^{H1}(z) \right].$$

- ▶ The result is given by

$$\begin{aligned} \frac{d\sigma^{(1),I}}{ds'} = \frac{d\sigma^{(0)}}{s} \frac{\alpha}{\pi} & \left[ \delta(1-z) \left( -2 + \frac{3}{2}L + 2\zeta_2 + 2(L-1) \ln(\Delta) \right) \right. \\ & \left. + \theta(1-z-\Delta) \frac{1+z^2}{1-z} (L-1) + \mathcal{O}\left(\frac{m_e^2}{s}\right) \right] \end{aligned}$$

with  $L = \ln(s/m_e^2)$ .

## The Differences between Two Calculations

ISR corrections have been **finally** calculated up to  $O(\alpha^2)$  in the asymptotic limit  $m_e^2/s \ll 1$  with two different techniques:

1. **Berends, Burgers, van Neerven (Nucl. Phys. B297 (1988))**:
  - ▶ Full calculation with massive electrons in the limit  $m_e^2 \ll s$  calculation in  $d = 4$  with soft-hard separation, including soft and virtual photons, hard bremsstrahlung, as well as fermion pair production.
  - ▶ Computational Technique:
    - Direct integration over the phase space in  $d = 4$  with soft-hard photon separator and photon mass to regulate the infrared.
    - Expansion in  $m_e^2 \ll s$  on integrand level (no details given).
2. **JB, De Freitas, van Neerven (Nucl. Phys. B855 (2012))**
  - ▶ Direct calculation of the asymptotic limit  $m_e^2 \ll s$  using massive light-cone operator matrix elements.
  - ▶ The technique is based on asymptotic factorization.  
Buza, Matiounine, Smith, Mignerone, van Neerven (Nucl.Phys. B472 (1996))
  - ▶ It was already used in Berends et al., but only for **the logarithmically enhanced terms**, claiming it works only at that level.

Kniehl, Krawczyk, Kühn, Stuart, Phys. Lett. B209 (1988):

Use as input: Baier et al. (1966):

(This is a **massless amplitude**, except the final state.)

$$\frac{d^2\sigma}{ds'ds''} = \sigma_0(s')\lambda^{1/2}(s, s', s'')P(s'', m_e^2) \left(\frac{\alpha}{\pi}\right)^2 \\ \times \left[ -2 + \frac{(s' + s'')^2 + s^2}{\lambda^{1/2}(s, s', s'')(s - s' - s'')} \ln \left[ \frac{s - s' - s'' + \lambda^{1/2}(s, s', s'')}{s - s' - s'' - \lambda^{1/2}(s, s', s'')} \right] \right]$$

- ▶ Berends et al. (1987) picked this expression up in their 1988 Erratum and agreed.
- ▶ **However, keeping  $m_e$  everywhere**, lead to our **2011 result**.
- ▶ This **Epiphany** has finally brought us on the right track:

**No neglect of  $m_e$  in all integrands!**

- ▶ The result by Kniehl et al. (1988), however, fully applies for ISR radiated pairs like  $\mu^+\mu^-$ ,  $\tau^+\tau^-$  and heavy quarks.
- ▶ Keeping  $m_e$  finite everywhere will lead to **monstrous expressions** at intermediary steps.

A calculation based on this has been impossible to anybody back in 1987, given the available computing resources and the lack of mathematical methods, known only since very recently, for a rigorous treatment.

# Factorization in the Asymptotic Region: The Method of Massive Operator Matrix Elements

We first consider the Method of massive OMEs.

In the asymptotic region the cross section factorizes

$$\frac{d\sigma_{ij}(s')}{ds'} = \frac{\sigma^{(0)}(s')}{s} \sum_{l,k} \Gamma_{l,i} \left( z, \frac{\mu^2}{m_e^2} \right) \otimes \tilde{\sigma}_{lk} \left( z, \frac{s'}{\mu^2} \right) \otimes \Gamma_{k,j} \left( z, \frac{\mu^2}{m_e^2} \right)$$

into

- massless cross sections  $\tilde{\sigma}_{ij} \left( z, \frac{s'}{\mu^2} \right)$   
Hamberg, van Neerven, Matsuura (Nucl. Phys. B 359 (1991))  
Harlander, Kilgore (Phys. Rev. Lett. 88 (2002))
- massive operator matrix elements  $\Gamma_{ij} \left( z, \frac{\mu^2}{m_e^2} \right)$ , which carry all mass dependence  
JB, De Freitas, van Neerven (Nucl.Phys. B855 (2012))

$\sigma^{(0)}(s')$  is the Born cross section and the convolution  $\otimes$  is given by

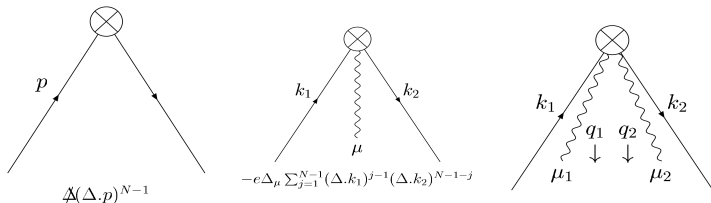
$$f(z) \otimes g(z) = \int_0^1 dz_1 \int_0^1 dz_2 f(z_1)g(z_2)\delta(z - z_1z_2).$$

The comparison between both calculations shows:

- ▶ the one-loop, i.e.  $O(\alpha)$ , agrees between both calculations
- ▶ the logarithmically enhanced terms at two-loops ( $O(\alpha^2)$ ) agree between both calculations
- ▶ the constant terms **do not agree**

⇒ breakdown of asymptotic factorization or errors?

# Factorization in the Asymptotic Region



$$\Gamma_{e^+e^+} = \Gamma_{e^-e^-} = \langle e | O_F^{\text{NS,S}} | e \rangle,$$

$$\Gamma_{e^+\gamma} = \Gamma_{e^-\gamma} = \langle \gamma | O_F^{\text{S}} | \gamma \rangle,$$

$$\Gamma_{\gamma e^+} = \Gamma_{\gamma e^-} = \langle e | O_V^{\text{S}} | e \rangle,$$

$$O_{F;\mu_1, \dots, \mu_N}^{\text{NS,S}} = i^{N-1} \text{S} [\bar{\psi} \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_N} \psi] - \text{traces},$$

$$O_{V;\mu_1, \dots, \mu_N}^{\text{S}} = 2i^{N-2} \text{S} [F_{\mu_1\alpha} D_{\mu_2} \dots D_{\mu_{N-1}} F_{\mu_N}^\alpha] - \text{traces}$$

- ▶ The technique has been used to derive deep-inelastic scattering (DIS) structure functions in the asymptotic limit  $Q^2 \gg m^2$  up to  $O(\alpha_s^3)$ .
- ▶ In the context of DIS **proven** to work at  $\alpha_s^2$  in the
  - non-singlet process  
Buza, Matiounine, Smith, van Neerven (Nucl.Phys. B485 (1997))  
Blümlein, Falcioni, De Freitas (Nucl.Phys. B910 (2016))
  - pure-singlet process  
Blümlein, De Freitas, Raab, Schönwald (Nucl.Phys. B945 (2019))

# The Renormalization Group Technique

We represent the observable in Mellin space transforming  $z = s'/s \in [0, 1]$  :

The differential scattering cross section  $\Sigma(z) = d\sigma_{ij}(z)/ds'$  is considered. This quantity reads in Mellin space

$$\mathbf{M}[\Sigma(z)](N) = \int_0^1 dz z^{N-1} \Sigma(z) .$$

In this representation the different Mellin convolutions to be performed in  $z$ -space simplify to ordinary products. The following representation is obtained

$$\frac{d\sigma_{ij}}{ds'}(N) = \frac{1}{s} \sigma^{(0)}(N) \sum_{l,k} \Gamma_{l,i} \left( N, \frac{\mu^2}{m^2} \right) \tilde{\sigma}_{lk} \left( N, \frac{s'}{\mu^2} \right) \Gamma_{k,j} \left( N, \frac{\mu^2}{m^2} \right) .$$

- ▶ Here  $\Gamma_{ji}$  denote massive operator matrix elements and  $\tilde{\sigma}_{lk}$  the massless Wilson coefficients, both being calculated in the  $\overline{\text{MS}}$  scheme.
- ▶  $\mu$  is the factorization scale, which cancels in the physical cross section.
- ▶ The initial state fermion mass dependence is solely encoded in  $\Gamma_{ji}$ .

# The Renormalization Group Technique

The solutions of these equations are

$$\Gamma_{ee} \left( N, a, \frac{\mu^2}{m^2} \right) = 1 + a \left[ -\frac{1}{2} \gamma_{ee}^{(0)} L + \Gamma_{ee}^{(0)} \right] + a^2 \left[ \left\{ \frac{1}{8} \gamma_{ee}^{(0)} \left( \gamma_{ee}^{(0)} - 2\beta_0 \right) + \frac{1}{8} \gamma_{e\gamma}^{(0)} \gamma_{\gamma e}^{(0)} \right\} L^2 + \frac{1}{2} \left\{ -\gamma_{ee}^{(1)} + 2\beta_0 \Gamma_{ee}^{(0)} - \gamma_{ee}^{(0)} \Gamma_{ee}^{(0)} - \gamma_{e\gamma}^{(0)} \Gamma_{\gamma e}^{(0)} \right\} L + \Gamma_{ee}^{(1)} \right] + O(a^3),$$

$$\tilde{\sigma}_{ee} \left( N, a, \frac{s'}{\mu^2} \right) = 1 + a \left[ -\frac{1}{2} \gamma_{ee}^{(0)} \lambda + \tilde{\sigma}_{ee}^{(0)} \right] + a^2 \left[ \left\{ \frac{1}{2} \gamma_{ee}^{(0)} \left( \gamma_{ee}^{(0)} + \beta_0 \right) + \frac{1}{4} \gamma_{e\gamma}^{(0)} \gamma_{\gamma e}^{(0)} \right\} \lambda^2 + 1 + \left\{ -\gamma_{ee}^{(1)} - \beta_0 \tilde{\sigma}_{ee}^{(0)} - \gamma_{ee}^{(0)} \tilde{\sigma}_{ee}^{(0)} - \gamma_{\gamma e}^{(0)} \tilde{\sigma}_{e\gamma}^{(0)} \right\} \lambda + \tilde{\sigma}_{ee}^{(1)} \right] + O(a^3),$$

$$\Gamma_{\gamma e} \left( N, a, \frac{\mu^2}{m^2} \right) = a \left[ -\frac{1}{2} \gamma_{\gamma e}^{(0)} L + \Gamma_{\gamma e}^{(0)} \right] + O(a^2)$$

$$\tilde{\sigma}_{e\gamma} \left( N, a, \frac{\mu^2}{m^2} \right) = a \left[ -\frac{1}{2} \gamma_{e\gamma}^{(0)} \lambda + \tilde{\sigma}_{e\gamma}^{(0)} \right] + O(a^2),$$

with the logarithms  $L = \ln \left( \frac{\mu^2}{m^2} \right)$  and  $\lambda = \ln \left( \frac{s'}{\mu^2} \right)$



# The Renormalization Group Technique

Introducing the **splitting functions in  $N$ -space**

$$P_{ij}^{(l)}(N) = \int_0^1 dz z^{N-1} P_{ij}^{(l)}(z) = -\gamma_{ij}^{(l)}(N)$$

one obtains

$$\begin{aligned} \frac{d\sigma_{e^+e^-}}{ds'} &= \frac{1}{s} \sigma^{(0)}(s) \left\{ 1 + a_0 \left[ P_{ee}^{(0)} \mathbf{L} + \left( \tilde{\sigma}_{ee}^{(0)} + 2\Gamma_{ee}^{(0)} \right) \right] \right. \\ &+ a_0^2 \left\{ \left[ \frac{1}{2} P_{ee}^{(0)} \otimes P_{ee}^{(0)} - \frac{\beta_0}{2} P_{ee}^{(0)} + \frac{1}{4} P_{e\gamma}^{(0)} \otimes P_{\gamma e}^{(0)} \right] \mathbf{L}^2 \right. \\ &+ \left. \left[ P_{ee}^{(1)} + P_{ee}^{(0)} \otimes \left( \tilde{\sigma}_{ee}^{(0)} + 2\Gamma_{ee}^{(0)} \right) - \beta_0 \tilde{\sigma}_{ee}^{(0)} + P_{\gamma e}^{(0)} \otimes \tilde{\sigma}_{e\gamma}^{(0)} + \Gamma_{\gamma e}^{(0)} \otimes P_{e\gamma}^{(0)} \right] \mathbf{L} \right. \\ &\left. \left. + \left( 2\Gamma_{ee}^{(1)} + \tilde{\sigma}_{ee}^{(1)} \right) + 2\Gamma_{ee}^{(0)} \otimes \tilde{\sigma}_{ee}^{(0)} + 2\tilde{\sigma}_{e\gamma}^{(0)} \otimes \Gamma_{\gamma e}^{(0)} + \Gamma_{ee}^{(0)} \otimes \Gamma_{ee}^{(0)} \right\} \right\} \end{aligned}$$

with

$$\mathbf{L} = \ln \left( \frac{s'}{m^2} \right) = \ln \left( \frac{s}{m^2} \right) + \ln(z); \quad \hat{\mathbf{L}} \equiv \ln(s/m^2) .$$

# The Renormalization Group Technique

It is convenient to represent the differential scattering cross section in terms of three contributions, the **flavor non-singlet** terms with a **single fermion line** (I), those with a **closed fermion line** (II), and the **pure-singlet terms** (III). These contributions are :

$$\begin{aligned}
 \frac{d\sigma_{e^+e^-}^{\text{I}}}{ds'} &= \frac{1}{s}\sigma^{(0)}(s)\left\{1 + a_0 \left[ P_{ee}^{(0)} \mathbf{L} + \left( \tilde{\sigma}_{ee}^{(0)} + 2\Gamma_{ee}^{(0)} \right) \right] \right. \\
 &\quad + a_0^2 \left\{ \frac{1}{2} P_{ee}^{(0)} \otimes P_{ee}^{(0)} \mathbf{L}^2 + \left[ P_{ee}^{(1),\text{I}} + P_{ee}^{(0)} \otimes \left( \tilde{\sigma}_{ee}^{(0)} + 2\Gamma_{ee}^{(0)} \right) \right] \mathbf{L} \right. \\
 &\quad \left. \left. + \left( 2\Gamma_{ee}^{(1),\text{I}} + \tilde{\sigma}_{ee}^{(1),\text{I}} \right) + 2\Gamma_{ee}^{(0)} \otimes \tilde{\sigma}_{ee}^{(0)} + \Gamma_{ee}^{(0)} \otimes \Gamma_{ee}^{(0)} \right\} \right\} \\
 \frac{d\sigma_{e^+e^-}^{\text{II}}}{ds'} &= \frac{1}{s}\sigma^{(0)}(s)a_0^2 \left\{ -\frac{\beta_0}{2} P_{ee}^{(0)} \mathbf{L}^2 + \left[ P_{ee}^{(1),\text{II}} - \beta_0 \tilde{\sigma}_{ee}^{(0)} \right] \mathbf{L} + \left( 2\Gamma_{ee}^{(1),\text{II}} + \tilde{\sigma}_{ee}^{(1),\text{II}} \right) \right\} \\
 \frac{d\sigma_{e^+e^-}^{\text{III}}}{ds'} &= \frac{1}{s}\sigma^{(0)}(s)a_0^2 \left\{ \frac{1}{4} P_{e\gamma}^{(0)} \otimes P_{\gamma e}^{(0)} \mathbf{L}^2 + \left[ P_{ee}^{(1),\text{III}} + P_{\gamma e}^{(0)} \otimes \tilde{\sigma}_{e\gamma}^{(0)} + \Gamma_{\gamma e}^{(0)} \otimes P_{e\gamma}^{(0)} \right] \mathbf{L} \right. \\
 &\quad \left. + \left( 2\Gamma_{ee}^{(1),\text{III}} + \tilde{\sigma}_{ee}^{(1),\text{III}} \right) + 2\tilde{\sigma}_{e\gamma}^{(0)} \otimes \Gamma_{\gamma e}^{(0)} \right\}
 \end{aligned}$$

- $\tilde{\sigma}_{ij}^{(k)}$  denotes the respective contribution of the massless Drell-Yan (DY) cross section.

## Different ingredients to the calculation :

- Splitting functions  $P_{ij}$  to  $O(\alpha^2)$

E.G. Floratos, D.A. Ross and C.T. Sachrajda, Nucl. Phys. B **129** (1977) 66 [Erratum-ibid. B **139** (1978) 545]; Nucl. Phys. B **152** (1979) 493;  
A. Gonzalez-Arroyo, C. Lopez and F.J. Yndurain, Nucl. Phys. B **153** (1979) 161;  
A. Gonzalez-Arroyo and C. Lopez, Nucl. Phys. B **166** (1980) 429;  
E.G. Floratos, C. Kounnas and R. Lacaze, Nucl. Phys. B **192** (1981) 417;  
G. Curci, W. Furmanski and R. Petronzio, Nucl. Phys. B **175** (1980) 27;  
W. Furmanski and R. Petronzio, Phys. Lett. B **97** (1980) 437;  
R. Hamberg and W.L. van Neerven, Nucl. Phys. B **379** (1992) 143;  
R.K. Ellis and W. Vogelsang, arXiv:hep-ph/9602356;  
S. Moch and J.A.M. Vermaseren, Nucl. Phys. B **573** (2000) 853;  
J. Ablinger *et al.*, Nucl. Phys. B **882** (2014) 263; Nucl. Phys. B **886** (2014) 733; Nucl. Phys. B **890** (2014) 48;  
Nucl. Phys. B **922** (2017) 1.

- massless Drell-Yan Cross Section  $\tilde{\sigma}_{ij}$  to  $O(\alpha^2)$

R. Hamberg, W.L. van Neerven and T. Matsuura, Nucl. Phys. B **359** (1991) 343 [E: B **644** (2002) 403];  
R.V. Harlander and W.B. Kilgore, Phys. Rev. Lett. **88** (2002) 201801.

- massive OMEs  $\Gamma_{ij}$  to  $O(\alpha^2)$   $\implies$  JB *et al.* 2011

## The 1-Loop OMEs

The  $O(\varepsilon^0)$  terms are

$$\begin{aligned}\Gamma_{ee}^{(0)}(x) &= -8\mathcal{D}_1(x) - 4\mathcal{D}_0(x) + 4\delta(1-x) + 2(1+x)[2\ln(1-x) + 1] \\ &= -4 \left[ \frac{1+x^2}{1-x} \left\{ \ln(1-x) + \frac{1}{2} \right\} \right]_+\end{aligned}$$

$$\Gamma_{e\gamma}^{(0)}(x) = 0$$

$$\Gamma_{\gamma e}^{(0)}(x) = -2 \frac{1+(1-x)^2}{x} [2\ln(x) + 1],$$

The linear term in  $\varepsilon \bar{\Gamma}_{ee}^{(0)}(x)$  reads

$$\begin{aligned}\bar{\Gamma}_{ee}^{(0)}(x) &= -4\mathcal{D}_2(x) - 4\mathcal{D}_1(x) - \zeta_2\mathcal{D}_0(x) - \left(4 + \frac{3}{4}\zeta_2\right)\delta(1-x) \\ &\quad + 2(1+x) \left[ \ln^2(1-x) + \ln(1-x) + \frac{1}{4}\zeta_2 \right] \\ &= -2 \left[ \frac{1+x^2}{1-x} \left\{ \ln^2(1-x) + \ln(1-x) + \frac{1}{4}\zeta_2 \right\} \right]_+ . \\ \mathcal{D}_m(x) &= \left( \frac{\ln^m(1-x)}{1-x} \right)_+\end{aligned}$$

# The Calculation of the Two-Loop Operator Matrix Elements



[1]



[2]



[3]



[4]



[5]



[8]



[11]



[13]



[14]



[17]



[18]

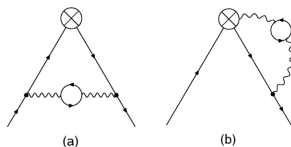


[20]

Two-loop diagrams contributing to the massive operator matrix element  $A_{ee}(N, \alpha)$ .  
The antisymmetric diagrams count twice.

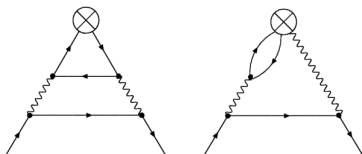
The result for the the matrix element  $\hat{F}_{ee}^{(1),I}$  is

$$\begin{aligned}
 & \left\{ \frac{1+3x^2}{1-x} \left[ 6\zeta_2 \ln(x) - 8 \ln(x) \text{Li}_2(1-x) - 4 \ln^2(x) \ln(1-x) \right] + \left( \frac{122}{3}x + 22 + \frac{32}{1-x} \right) \zeta_2 + (8 - 112\zeta_2) \mathcal{D}_1(x) \right. \\
 & + 16 \frac{1+x^2}{1-x} \left[ 2\text{Li}_3(-x) - \ln(x) \text{Li}_2(-x) \right] + \frac{80}{3(1-x)} + 56(1+x)\zeta_2 \ln(1-x) + (16 - 52\zeta_2 + 128\zeta_3) \mathcal{D}_0(x) \\
 & + \left( \frac{22}{3}x + 32 + \frac{64}{3(1-x)^2} - \frac{51}{1-x} - \frac{16}{3(1-x)^3} \right) \ln^2(x) - (92 + 20x) \ln^2(1-x) + 14(x-2) \ln(1-x) + 120\mathcal{D}_2(x) \\
 & + \left( \frac{178}{3} - 36x + \frac{64}{3(1-x)^2} - \frac{140}{3(1-x)} - \frac{48}{1+x} \right) \ln(x) - \frac{1}{3}(1+x) \ln^3(x) + 4 \frac{x^2 - 8x - 6}{1-x} \ln(x) \ln(1-x) \\
 & - 2 \frac{1+17x^2}{1-x} \ln(x) \ln^2(1-x) - \frac{112}{3}(1+x) \ln^3(1-x) + 32 \frac{1+x}{1-x} \left[ \ln(x) \ln(1+x) + \text{Li}_2(-x) \right] - 22x - \frac{62}{3} \\
 & - 4 \frac{13x^2+9}{1-x} \text{S}_{1,2}(1-x) + 4 \frac{5-11x^2}{1-x} \left[ \ln(1-x) \text{Li}_2(1-x) - \text{Li}_3(1-x) - 2\zeta_3 \right] + \frac{4(16x^2-10x-27)}{3(1-x)} \text{Li}_2(1-x) \\
 & + \frac{224}{3} \mathcal{D}_3(x) + \left[ \frac{433}{8} - \frac{67}{45} \pi^4 + \left( \frac{37}{2} - 48 \ln(2) \right) \zeta_2 + 58\zeta_3 \right] \delta(1-x) \left. \right\} + (-1)^n \left\{ \frac{2(1-x)(45x^2+74x+45)}{3(1+x)^2} \right. \\
 & + \frac{2(9+12x+30x^2-20x^3-15x^4)}{3(1+x)^3} \ln(x) + \frac{4(x^2+10x-3)}{3(1+x)} (\zeta_2 + 2\text{Li}_2(-x) + 2 \ln(x) \ln(1+x)) \\
 & + \frac{1+x^2}{1+x} \left[ 36\zeta_3 - 24\zeta_2 \ln(1+x) + 8\zeta_2 \ln(x) - \frac{2}{3} \ln^3(x) + 40\text{Li}_3(-x) - 4 \ln^2(x) \ln(1+x) - 24 \ln(x) \ln^2(1+x) \right. \\
 & \left. - 24 \ln(x) \text{Li}_2(-x) - 48 \ln(1+x) \text{Li}_2(-x) - 8 \ln(x) \text{Li}_2(1-x) - 16\text{S}_{1,2}(1-x) - 48\text{S}_{1,2}(-x) \right] \\
 & \left. - \frac{16(x^4+12x^3+12x^2+8x+3)}{3(1+x)^3} \text{Li}_2(1-x) + 4x \frac{1-x-5x^2+x^3}{(1+x)^3} \ln^2(x) \right\}
 \end{aligned}$$



The result for  $\hat{\Gamma}_{ee}^{(1),II}$  is

$$\begin{aligned}
 \hat{\Gamma}_{ee}^{(1),II} = & \frac{76}{27}x - \frac{572}{27} - \left(12x + \frac{4}{3} + \frac{8}{1-x}\right) \ln(x) + \frac{128}{9(1-x)^2} + \frac{80}{27(1-x)} - \frac{64}{9(1-x)^3} \\
 & - \frac{32}{9} \left( \frac{1}{(1-x)^2} - \frac{5}{(1-x)^3} + \frac{2}{(1-x)^4} \right) \ln(x) + \frac{16}{3}(1+x) \left( \ln(1-x) + \ln^2(1-x) \right) \\
 & - \frac{2(1+x^2)}{3(1-x)} \ln^2(x) + \left( \frac{224}{27} - \frac{8}{3}\zeta_2 \right) \mathcal{D}_0(x) + \frac{4}{3}(1+x)\zeta_2 - \frac{32}{3} (\mathcal{D}_1(x) + \mathcal{D}_2(x)) \\
 & + \left( \frac{8}{3}\zeta_3 + 10\zeta_2 - \frac{1411}{162} \right) \delta(1-x)
 \end{aligned}$$



The result for  $\hat{\Gamma}_{ee}^{(1),III}$  is

(a)

(b)

$$\begin{aligned} \hat{\Gamma}_{ee}^{(1),III} &= \frac{2}{x}(1-x)(4x^2 + 13x + 4)\zeta_2 + \frac{1}{3x}(8x^3 + 135x^2 + 75x + 32)\ln^2(x) \\ &+ \left[ \frac{304}{9x} - \frac{80}{9}x^2 - \frac{32}{3}x + 108 - \frac{32}{1+x} - \frac{64(1+2x)}{3(1+x)^3} \right] \ln(x) - \frac{224}{27}x^2 \\ &+ 16\frac{1-x}{3x}(x^2 + 4x + 1)[2\ln(x)\ln(1+x) - \text{Li}_2(1-x) + 2\text{Li}_2(-x)] \\ &+ (1+x) \left[ 4\zeta_2 \ln(x) + \frac{14}{3}\ln^3(x) - 32\ln(x)\text{Li}_2(-x) - 16\ln(x)\text{Li}_2(x) + 64\text{Li}_3(-x) \right. \\ &\left. + 32\text{Li}_3(x) + 16\zeta_3 \right] - \frac{182}{3}x + 50 - \frac{32}{1+x} + \frac{800}{27x} + \frac{64}{3(1+x)^2} \end{aligned}$$

The first moment vanishes for all three contributions  $\hat{\Gamma}_{ee}^{(1),I}$ ,  $\hat{\Gamma}_{ee}^{(1),II}$  and  $\hat{\Gamma}_{ee}^{(1),III}$ .

→ Fermion number conservation is satisfied.



# The Scattering Cross Section

The 2-loop corrections to the process  $e^+e^- \rightarrow Z^0$  can be organized in the following form :

$$\frac{d\sigma_{e^+e^-}}{ds'} = \frac{1}{s} \sigma^{(0)}(s) \left\{ 1 + a_0 \left[ T_{11} \hat{\mathbf{L}} + T_{10} \right] + a_0^2 \left[ T_{22} \hat{\mathbf{L}}^2 + T_{21} \hat{\mathbf{L}} + T_{20} \right] \right\}$$

• Universal Corrections :  $T_{ii}(z)$   $\implies$  depend on LO splitting functions and  $\beta_0$

$$T_{11} = 8\mathcal{D}_0(z) - 4(1+z) + 6\delta(1-z) = 4 \left[ \frac{1+z^2}{1-z} \right]_+$$

$$T_{22} = \left\{ 64\mathcal{D}_1(z) + 48\mathcal{D}_0(z) + (18 - 32\zeta_2)\delta(1-z) - 32 \frac{\ln(z)}{1-z} - 32(1+z) \ln(1-z) + 24(1+z) \ln(z) - 8(5+z) \right\}_I$$
$$+ \frac{2}{3} \left\{ 8\mathcal{D}_0(z) - 4(1+z) + 6\delta(1-z) \right\}_{II}$$
$$+ 16 \left\{ \frac{1}{2}(1-z) \ln(z) + \frac{1}{4}(1-z) + \frac{1}{3} \frac{1}{3z} (1-z^3) \right\}_{III} .$$

# The Cross Section at $O(\alpha)$ and the Logarithmic 2-Loop Contributions

- $O(\alpha)$  Term :  $T_{10}(z)$   $\implies$  depend on LO OME + LO DY

$$T_{10} = -4 \left[ \frac{1+z^2}{1-z} \right]_+ + 2(4\zeta_2 - 1)\delta(1-z)$$
$$T_{11}\hat{\mathbf{L}} + T_{10} = P_{ee}^{(0)}(z) [\hat{\mathbf{L}} - 1] + 2(4\zeta_2 - 1)\delta(1-z).$$

## Complete 1-Loop Result.

- $O(\alpha^2\hat{\mathbf{L}})$  Terms :  $T_{21}(z)$   $\implies$  depend on LO,NLO splitting fcts., LO OME + LO DY

Contributions to the three main processes I-III :

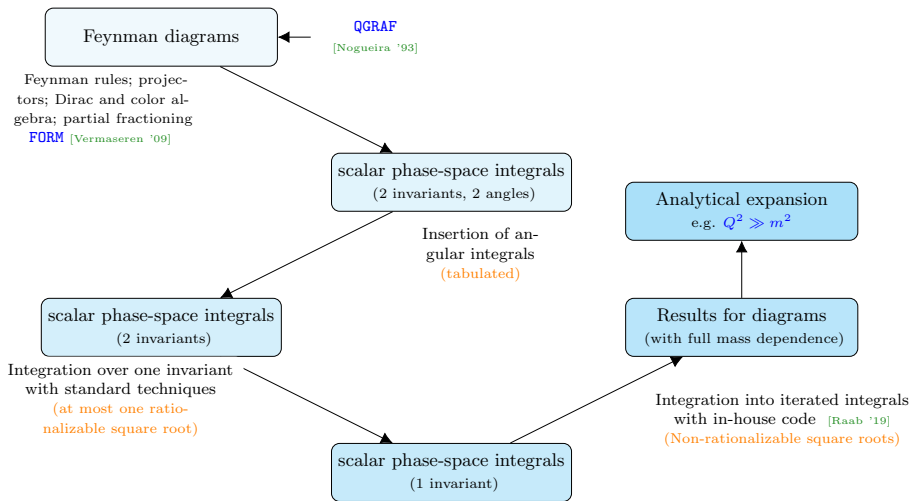
$$T_{21}^I = 16 \left\{ -8\mathcal{D}_1(z) - (7 - 4\zeta_2)\mathcal{D}_0(z) + \left( -\frac{45}{16} + \frac{11}{2}\zeta_2 + 3\zeta_3 \right) \delta(1-z) \right. \\ \left. + \left( \frac{1+z^2}{1-z} \right) \left[ \ln(z)\ln(1-z) - \ln^2(z) + \frac{11}{4}\ln(z) \right] \right. \\ \left. + (1+z) \left[ 4\ln(1-z) + \frac{1}{4}\ln^2(z) - \frac{7}{4}\ln(z) - 2\zeta_2 \right] - \ln(z) + 3 + 4z \right\}$$

$$\begin{aligned}
 T_{21}^{\text{II}} &= 16 \left\{ \frac{4}{3} \mathcal{D}_1(z) - \frac{10}{9} \mathcal{D}_0(z) - \frac{17}{12} \delta(1-z) \right. \\
 &\quad \left. - \frac{2}{3} \frac{\ln(z)}{1-z} - \frac{1}{3} (1+z) [2 \ln(1-z) - \ln(z)] - \frac{1}{9} + \frac{11}{9} z \right\} \\
 T_{21}^{\text{III}} &= 16 \left\{ (1+z) \left[ 2 \text{Li}_2(1-z) - \ln^2(z) + 2 \ln(z) \ln(1-z) \right] \right. \\
 &\quad + \left( \frac{4}{3} \frac{1}{z} + 1 - z - \frac{4}{3} z^2 \right) \ln(1-z) - \left( \frac{2}{3} \frac{1}{z} + 1 - \frac{1}{2} z - \frac{4}{3} z^2 \right) \ln(z) \\
 &\quad \left. - \frac{8}{9} \frac{1}{z} - \frac{8}{3} + \frac{8}{3} z + \frac{8}{9} z^2 \right\}
 \end{aligned}$$

Up to this point, we find agreement with Berends et al. (1988), but disagree for  $T_{20}$ .

## The $\mathcal{O}(\alpha^2)$ Corrections by Direct Calculation

- ▶ In Berends et al. the  $\mathcal{O}(\alpha^2)$  corrections have been split up into four distinct processes:
  - Process I, photon radiation
  - Process II, non-singlet fermion pair production
  - Process III, pure-singlet fermion pair production
  - Process IV, interference between non-singlet and pure-singlet fermion pair production
- ▶ In the calculation of Blümlein et al. (Nucl. Phys. B855 (2012)) process I and IV had to be treated combined due to the nature of the OMEs, which avoids cutting techniques.
- ▶ We have to distinguish between vector and axial/couplings of the Z-boson.  
⇒ We thoroughly work in  $d = 4$  dimensions.
- ▶ We have to recalculate and to add contributions due to diagrams not having been considered before.



- ▶ We want to use iterated integrals so we can work in a **differential field**.

$$H_{w_1, \dots, w_n}(x) = \int_0^x dt f_{w_n}(t) H_{w_1, \dots, w_{n-1}}(x), \quad \tilde{H}_{w_1, \dots, w_n}(x) = \int_x^1 dt f_{w_n}(t) H_{w_1, \dots, w_{n-1}}(x).$$

- ▶ The steps to transform the last integrand to iterated integrals include:
  - Express all logarithms and polylogarithms in terms of iterated integrals evaluated at the last integration variable through linear differential equations.
  - Find relations between the occurring letters and square roots to get rid of redundancies.
  - Compactify the integrand expressed in terms of iterated integrals as far as possible.
    - Since we express everything in linearly independent quantities, the complexity of the last integral can be drastically reduced in this step.
- ▶ The same technique has been successfully applied to calculate the full mass dependence of the pure-singlet structure functions in deep-inelastic-scattering.
- ▶ In total we need **37 letters** to express the contributions due to fermion pair production.

# The Letters:

$$v_1 = \frac{1}{\sqrt{1-4t}\sqrt{16t^2-8(1+z)t+(1-z)^2}}$$

$$v_2 = \frac{1}{t\sqrt{1-4t}\sqrt{16t^2-8(1+z)t+(1-z)^2}}$$

$$v_3 = \frac{1}{\sqrt{1-4t}(4t-(1+x))\sqrt{16t^2-8(1+z)t+(1-z)^2}}$$

$$v_4 = \frac{1}{t\sqrt{1-t}}$$

$$d_1 = \frac{1}{\sqrt{1-t}\sqrt{16\rho^2-8\rho(1+z)t+(1-z)^2t^2}}$$

$$d_2 = \frac{t}{\sqrt{1-t}\sqrt{16\rho^2-8\rho(1+z)t+(1-z)^2t^2}}$$

$$d_3 = \frac{1}{t\sqrt{1-t}\sqrt{16\rho^2-8\rho(1+z)t+(1-z)^2t^2}}$$

$$d_4 = \frac{1}{(16\rho^2+(4z-8\rho(1+z))t+(1-z)^2t^2)\sqrt{1-t}\sqrt{16\rho^2-8\rho(1+z)t+(1-z)^2t^2}}$$

$$d_5 = \frac{1}{(16\rho^2+(4z-8\rho(1+z))t+(1-z)^2t^2)\sqrt{1-t}\sqrt{16\rho^2-8\rho(1+z)t+(1-z)^2t^2}}$$

$$d_6 = \frac{1}{(16\rho^2+(4z-8\rho(1+z))t+(1-z)^2t^2)\sqrt{16\rho^2-8\rho(1+z)t+(1-z)^2t^2}}$$

$$d_7 = \frac{t}{(16\rho^2+(4z-8\rho(1+z))t+(1-z)^2t^2)\sqrt{16\rho^2-8\rho(1+z)t+(1-z)^2t^2}}$$

$$d_8 = \frac{1-z}{(4\rho-(1-z)t)\sqrt{1-t}}$$

$$d_9 = \frac{1}{(16\rho^2+4(z-2\rho(1+z))t+(1-z)^2t^2)\sqrt{1-t}}$$

$$d_{10} = \frac{1}{(16\rho^2+4(z-2\rho(1+z))t+(1-z)^2t^2)\sqrt{1-t}}$$

$$d_{11} = \frac{1}{t\sqrt{16\rho^2-8\rho(1+z)t+(1-z)^2t^2}}$$

$$d_{12} = \frac{1}{16\rho^2+4(z-2\rho(1+z))t+(1-z)^2t^2}$$

$$d_{13} = \frac{t}{16\rho^2+4(z-2\rho(1+z))t+(1-z)^2t^2}$$

$$d_{14} = \frac{1}{t(1-z)-4\rho}$$

$$d_{15} = \frac{1}{\sqrt{1-t}(t(1-z)-4\rho)}$$

$$d_{16} = \frac{1}{\sqrt{t(1-t)}\sqrt{t(1-z)^2-16\rho^2}}$$

$$d_{17} = \frac{1}{\sqrt{t(1-t)}(t(1-z)-4\rho)\sqrt{t(1-z)^2-16\rho^2}}$$

$$d_{18} = \frac{1}{\sqrt{t}\sqrt{t(1-z)^2-16\rho^2}}$$

$$d_{19} = \frac{1}{\sqrt{t}(t(1-z)-4\rho)\sqrt{t(1-z)^2-16\rho^2}}$$

$$d_{20} = \frac{1}{\sqrt{t^2(1-z)^2-8\rho t(1+z)+16\rho^2}}$$

$$d_{21} = \frac{1}{\sqrt{1-t}\sqrt{t^2(1-z)^2-8\rho t(1+z)+16\rho^2}}$$

$$d_{22} = \frac{\sqrt{t}}{\sqrt{t(1-z)^2-16\rho^2}\sqrt{t^2(1-z)^2-8\rho t(1+z)+16\rho^2}}$$

$$d_{23} = \frac{\sqrt{t}}{\sqrt{t(1-z)^2-16\rho^2}(t^2(1-z)^2-8\rho t(1+z)+4tz+16\rho^2)}$$

$$d_{24} = \frac{1}{(t^2(1-z)^2-8\rho t(1+z)t+4tz+16\rho^2)\sqrt{t^2(1-z)^2-8\rho t(1+z)+16\rho^2}}$$

$$d_{25} = \frac{1}{(t^2(1-z)^2-8\rho t(1+z)t+4tz+16\rho^2)\sqrt{t^2(1-z)^2-8\rho t(1+z)+16\rho^2}}$$

$$d_{26} = \frac{1}{\sqrt{1-t}(t^2(1-z)^2-8\rho t(1+z)t+4tz+16\rho^2)\sqrt{t^2(1-z)^2-8\rho t(1+z)+16\rho^2}}$$

$$d_{27} = \frac{1}{\sqrt{1-t}(t^2(1-z)^2-8\rho t(1+z)t+4tz+16\rho^2)\sqrt{t^2(1-z)^2-8\rho t(1+z)+16\rho^2}}$$

$$d_{28} = \frac{1}{\sqrt{t}\sqrt{t(-1+z)^2-16\rho^2}\sqrt{t^2(1-z)^2-8\rho t(1+z)+16\rho^2}}$$

$$d_{29} = \frac{1}{\sqrt{t}\sqrt{t(1-z)^2-16\rho^2}(t^2(1-z)^2-8\rho t(1+z)t+4tz+16\rho^2)}$$

$$d_{30} = \frac{1}{\sqrt{t}\sqrt{t(1-z)^2-16\rho^2}(t^2(1-z)^2-8\rho t(1+z)t+4tz+16\rho^2)\sqrt{t^2(1-z)^2-8\rho t(1+z)+16\rho^2}}$$

$$d_{31} = \frac{1}{\sqrt{t(1-z)^2-16\rho^2}(t^2(1-z)^2-8\rho t(1+z)t+4tz+16\rho^2)\sqrt{t^2(1-z)^2-8\rho t(1+z)+16\rho^2}}$$

$$d_{32} = \frac{1}{t\sqrt{1-t}\sqrt{t^2(1-z)^2-8\rho t(1+z)+16\rho^2}}$$

$$d_{33} = \frac{1}{\sqrt{1-t}\sqrt{t^2(1-z)^2-8\rho t(1+z)+16\rho^2}}$$

$$d_1 = \frac{1}{\sqrt{1-t}\sqrt{16\rho^2 - 8\rho(1+z)t + (1-z)^2t^2}}$$

$$d_2 = \frac{t}{\sqrt{1-t}\sqrt{16\rho^2 - 8\rho(1+z)t + (1-z)^2t^2}}$$

$$d_3 = \frac{1}{t\sqrt{1-t}\sqrt{16\rho^2 - 8\rho(1+z)t + (1-z)^2t^2}}$$

$$d_4 = \frac{1}{(16\rho^2 + (4z - 8\rho(1+z))t + (1-z)^2t^2)\sqrt{1-t}\sqrt{16\rho^2 - 8\rho(1+z)t + (1-z)^2t^2}}$$

$$d_5 = \frac{t}{(16\rho^2 + (4z - 8\rho(1+z))t + (1-z)^2t^2)\sqrt{1-t}\sqrt{16\rho^2 - 8\rho(1+z)t + (1-z)^2t^2}}$$

$$d_{16} = \frac{1}{\sqrt{t(1-t)}\sqrt{t(1-z)^2 - 16\rho^2}}$$

$$d_{17} = \frac{1}{\sqrt{t(1-t)}(t(1-z) - 4\rho)\sqrt{t(1-z)^2 - 16\rho^2}}$$

$$d_{21} = \frac{1}{\sqrt{1-t}\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}}$$

$$d_{22} = \frac{\sqrt{t}}{\sqrt{t(1-z)^2 - 16\rho^2}\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}}$$

$$d_{26} = \frac{1}{\sqrt{1-t}(t^2(1-z)^2 - 8\rho(1+z)t + 4tz + 16\rho^2)\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}}$$

$$d_{27} = \frac{t}{\sqrt{1-t}(t^2(1-z)^2 - 8\rho(1+z)t + 4tz + 16\rho^2)\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}}$$

$$d_{28} = \frac{1}{\sqrt{t}\sqrt{t(-1+z)^2 - 16\rho^2}\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}}$$

$$d_{30} = \frac{1}{\sqrt{t}\sqrt{t(1-z)^2 - 16\rho^2}(t^2(1-z)^2 - 8\rho(1+z)t + 4tz + 16\rho^2)\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}}$$

$$d_{31} = \frac{\sqrt{t}}{\sqrt{t(1-z)^2 - 16\rho^2}(t^2(1-z)^2 - 8\rho(1+z)t + 4tz + 16\rho^2)\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}}$$

$$d_{32} = \frac{1}{t\sqrt{1-t}\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}}$$

$$d_{33} = \frac{t}{\sqrt{1-t}\sqrt{t^2(1-z)^2 - 8\rho t(1+z) + 16\rho^2}}$$

- ▶ 16 of these letters introduce elliptic structures, since multiple square roots cannot be rationalized at once.

$\rho = m_e^2/s$ ,  $z = s'/s$ ,  $t$  - integration variable.



# The Size of the Calculation

## Size of amplitudes:

process I	10Gb
process II	25kb
process III	56kb
process IV	124kb

## Computation time:

	Reduction to Basis	Integration	
process I		30h	several months of code design
process II	1 day	min's	
process III	1 month	2h	
process IV	2 months	5h	

## The Non-Singlet Case

$$\begin{aligned}
 \frac{d\sigma^{(2),\text{II}}(z, \rho)}{ds'} &= \frac{\sigma^{(0)}(s')}{s} a^2 \left\{ \frac{64}{3} z(1-z)(1+z-4\rho) \tilde{\text{H}}_{v_4, d_7} + \frac{256}{3} z\rho(1+z-4\rho) \tilde{\text{H}}_{v_4, d_6} \right. \\
 &+ \frac{128z(1-4\rho^2)(1-z+2\rho)(1-z-4\rho)}{3(1-z)^2} \tilde{\text{H}}_{d_8, d_7} \\
 &+ \frac{512z\rho(1-4\rho^2)(1-z+2\rho)(1-z-4\rho)}{3(1-z)^3} \tilde{\text{H}}_{d_8, d_6} \\
 &+ \frac{16}{9(1-z)^2} \left[ (1+z)^2(4-9z+4z^2) + 2(9-16z+13z^2-2z^3)\rho + 32\rho^2 \right] \tilde{\text{H}}_{d_2} \\
 &+ \frac{512z\rho}{9(1-z)^4} \left[ 3(1-z)^4 z - (1-z)^3(4+z^2)\rho - 2(9-29z+38z^2-17z^3+3z^4)\rho^2 \right. \\
 &- 4(2-z)(3+6z-5z^2)\rho^3 + 16(7-8z+9z^2)\rho^4 + 128(3-z)\rho^5 \left. \right] \tilde{\text{H}}_{d_4} \\
 &- \frac{16}{9(1-z)^4} \left[ 3-34z+129z^2-212z^3+129z^4-34z^5+3z^6+8(2-16z+9z^2 \right. \\
 &+ 4z^3-5z^4+2z^5)\rho + 16z(12-13z+18z^2-z^3)\rho^2 + 32(1+22z-7z^2)\rho^3 \left. \right] \tilde{\text{H}}_{d_1} \\
 &- \frac{128z}{9(1-z)^4} \left[ 1+7z-47z^2+86z^3-47z^4+7z^5+z^6-2(7-55z+54z^2 \right. \\
 &+ 16z^3-17z^4+3z^5)\rho - 4(39-16z+16z^2+4z^3+5z^4)\rho^2 \\
 &+ 16(8-23z+22z^2+9z^3)\rho^3 + 128(7+2z-z^2)\rho^4 \left. \right] \tilde{\text{H}}_{d_5} - \frac{64}{3} (2z+(1-z)\rho) \tilde{\text{H}}_{d_3} \\
 &+ \left[ \frac{16}{3\sqrt{1-4\rho}} (1+z-4\rho) \tilde{\text{H}}_{v_4} + \frac{32(1-4\rho^2)(1-z+2\rho)(1-z-4\rho)}{3(1-z)^3\sqrt{1-4\rho}} \tilde{\text{H}}_{d_8} \right] \\
 &\times \ln \left( \frac{1-z-4\rho-\sqrt{1-4\rho}\sqrt{(1-z)^2-8(1+z)\rho+16\rho^2}}{1-z-4\rho+\sqrt{1-4\rho}\sqrt{(1-z)^2-8(1+z)\rho+16\rho^2}} \right) \left. \right\}
 \end{aligned}$$

Complete cross section without any approximation. Note the new iterated integrals, which are found using Risch-algorithm techniques.

## The Non-Singlet Case

- ▶ The explicit expansion of the analytical result in the limit  $m_e^2 \ll s$  gives

$$\begin{aligned} \frac{d\sigma^{(2),II}(z)}{ds'} &= \frac{\sigma^{(0)}(s')}{s} \left(\frac{\alpha}{4\pi}\right)^2 \left\{ \frac{8}{3} \frac{1+z^2}{1-z} L^2 - \left[ \frac{16}{9} \frac{11-12z+11z^2}{1-z} - \frac{16}{3} \frac{1+z^2}{1-z} \ln(z) \right. \right. \\ &\quad \left. \left. + \frac{32}{3} \frac{1+z^2}{1-z} \ln(1-z) \right] L + \frac{32}{9(1-z)^3} (7-13z+8z^2-13z^3+7z^4) \right. \\ &\quad \left. - \frac{16z}{9(1-z)^4} (3-36z+94z^2-72z^3+19z^4) \ln(z) - \frac{8z^2}{3(1-z)} \ln^2(z) \right. \\ &\quad \left. - \left( \frac{32}{9} \frac{11-12z+11z^2}{1-z} + \frac{16}{3} \frac{2+z^2}{1-z} \ln(z) \right) \ln(1-z) + \frac{32}{3} \frac{1+z^2}{1-z} \ln^2(1-z) \right. \\ &\quad \left. + \frac{16z^2}{3(1-z)} \text{Li}_2(z) - \frac{16(2+3z^2)}{3(1-z)} \zeta_2 \right\} + \mathcal{O}\left(\frac{m_e^2}{s} L^2\right), \end{aligned}$$

with  $L = \ln(m_e^2/s)$ .

- ▶ The result contains higher denominator powers which have **not been obtained** by Berends et al.

$$\begin{aligned} \delta_{II} &= -\frac{128}{9} \left[ 3 + \frac{1}{(1-z)^3} - \frac{2}{(1-z)^2} - 2z \right] - 16 \left[ 1 + \frac{5z}{3} + \frac{8}{9} \frac{1}{(1-z)^4} \right. \\ &\quad \left. - \frac{20}{9} \frac{1}{(1-z)^3} + \frac{4}{9} \frac{1}{(1-z)^2} \right] \ln(z) + \frac{8}{3} \frac{1+z^2}{1-z} \left[ \frac{10}{9} - \frac{14}{3} \ln(z) - \ln^2(z) \right]. \end{aligned}$$

## 2 photon emission:

- ▶  $T_2^{S2}$ : both emitted photons are soft ✓
- ▶  $T_2^{V2}$ : both photons are virtual ✓
- ▶  $T_2^{S1V1}$ : one photon is soft, one is virtual ✓
- ▶  $T_2^{S1H1}$ : one photon is soft, one is hard ✓
- ▶  $T_2^{V1H1}$ : one photon is virtual, one is hard: **disagreement**
- ▶  $T_2^{H2}$ : both emitted photons are hard, ✓

Here and in the following we only report the vector-case.

There are differences in the axial-vector case (not clear from Berends et al.)

Since we can work in 4-dimensions (only Abelian couplings) we can treat  $\gamma_5$  without a further finite renormalization.

- ▶ We find the difference:

$$\delta_I = -8 + \frac{100}{33} z \ln^2(z) - 8 \ln(1-z) + 4(1-z) \ln^2(1-z) - \frac{8(2-z)z}{1-z} \text{Li}_2(z) + \frac{8(2-2z+z^2)}{1-z} \zeta_2,$$

## Process III: Pure Singlet

$$\begin{aligned}
 \delta_{III} = & \frac{160}{3} - \frac{32}{z} + \frac{128}{3(1+z)^2} - \frac{64}{1+z} + 96(1+z)\zeta_3 - \left[ 52(1-z) + \frac{64}{3z}(1-z^3) \right] \ln^2(z) \\
 & - \frac{56}{3}(1+z) \ln^3(z) + \left[ 24(1-z) + 16(1+z) \ln(z) \right] \zeta_2 + \ln(z) \left[ \frac{104}{3} - \frac{32}{z} + \frac{128}{3(1+z)^3} \right. \\
 & - \frac{256}{3(1+z)^2} - \frac{64}{1+z} + 64 \left( 1 - z + \frac{1-z^3}{3z} \right) \ln(1+z) \left. \right] - \left[ 40(1-z) + \frac{64}{3z}(1-z^3) \right. \\
 & \left. + 48(1+z) \ln(z) \right] \text{Li}_2(1-z) + 64 \left[ 1 - z + \frac{1}{3z}(1-z^3) - (1+z) \ln(z) \right] \text{Li}_2(-z) \\
 & + 128(1+z) \text{Li}_3(-z) - 96(1+z) S_{1,2}(1-z) + 2\delta_{\text{interf}}^{\text{PS}}, \\
 \frac{d\sigma_{\text{interf}}^{(2),\text{III}}}{ds'} = & \frac{\sigma^{(0)}(s')}{s} \left( \frac{\alpha}{4\pi} \right)^2 \left\{ -160(1-z) - \left[ 16(5+4z) - 80(1+z)H_{-1} + \frac{48(2+2z+z^2)}{z}H_{-1}^2 \right] H_0 \right. \\
 & - \left[ 52z - \frac{40(2+2z+z^2)}{z}H_{-1} \right] H_0^2 - \frac{16}{3}zH_0^3 + \left[ 8(5-4z)H_0 - \frac{8(4-6z+3z^2)}{z}H_0^2 \right] H_1 \\
 & - \frac{4(4-6z+3z^2)}{z}H_0H_1^2 - \left[ 8(5-4z) - \frac{8(8-2z+5z^2)}{z}H_0 - \frac{8(4-6z+3z^2)}{z}H_1 \right] H_{0,1} \\
 & - \left[ 80(1+z) + \frac{32(5+2z^2)}{z}H_0 - \frac{96(2+2z+z^2)}{z}H_{-1} \right] H_{0,-1} - \frac{32(2+2z+z^2)}{z}H_{0,0,1} \\
 & + \frac{16(10-10z+3z^2)}{z}H_{0,0,-1} - \frac{8(4-6z+3z^2)}{z}H_{0,1,1} - \frac{96(2+2z+z^2)}{z}H_{0,-1,-1} \\
 & \left. + \left[ 8(10+z) + 160H_0 - \frac{8(4-6z+3z^2)}{z}H_1 - \frac{48(2+2z+z^2)}{z}H_{-1} \right] \zeta_2 + 32(5+z)\zeta_3 \right\} \\
 & + \mathcal{O}\left(\frac{m^2}{s}\right)
 \end{aligned}$$

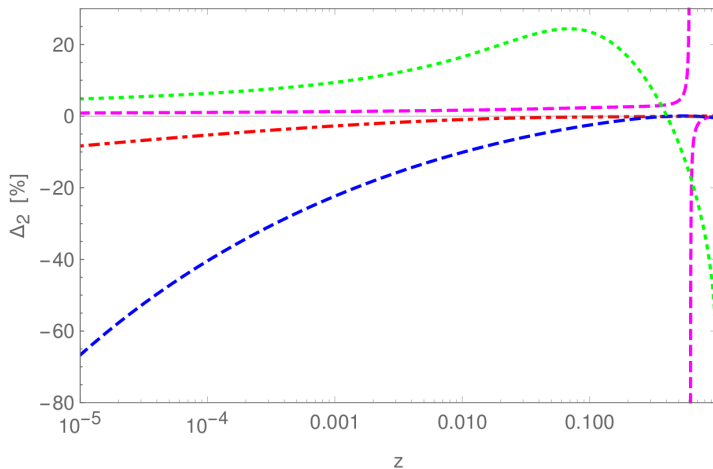
⇒ First calculated by A.N. Schellekens (Thesis, Nijmegen, 1981)

- We find the difference:

$$\begin{aligned} \delta_{IV} = & \frac{2(53 + 994z + 32z^2 + 742z^3 - 85z^4 - 8z^5)}{9(1-z)(1+z)^2} - 8 \left[ \frac{1 - 14z - 56z^2 + 78z^3 - 25z^4}{(1-z^2)^2} \right. \\ & \left. + \frac{1+z^2}{1-z} \ln(z) \right] \zeta_2 - \frac{8z(13 + 12z^2 - 20z^3 + 3z^4)}{(1-z^2)^2} \ln^2(z) \\ & + 16 \left[ \frac{1-z+7z^2-3z^3}{(1+z)^2} + \frac{7+3z^2}{2(1-z)} \ln(z) \right] \text{Li}_2(1-z) + \left[ \frac{32(1+5z-4z^2)}{(1-z)^2} \ln(1+z) \right. \\ & \left. - \frac{16(4-7z-6z^2-128z^3+2z^4-9z^5)}{3(1-z)^2(1+z)^3} \right] \ln(z) + \frac{32(1+5z-4z^2)}{(1-z)^2} \text{Li}_2(-z), \end{aligned}$$

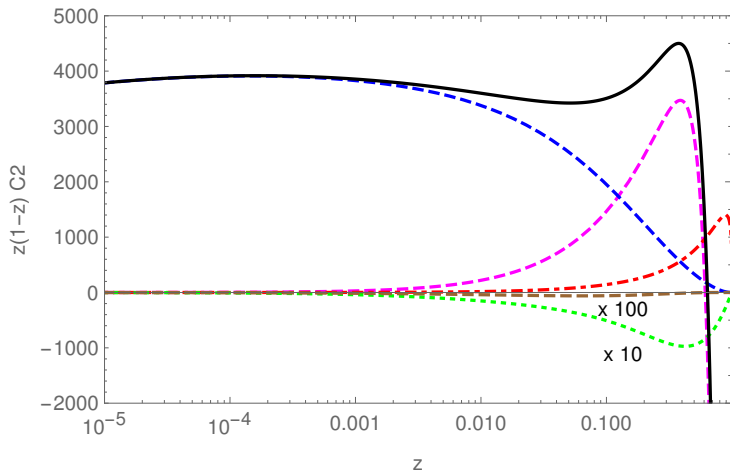
⇒ Our results **agree** with the ones obtained in [JB, De Freitas, van Neerven, 2011](#).

## Numerical Illustrations



- ▶ Relative deviation of the photon (I), non-singlet (II), pure-singlet (III) and their interference (IV) contribution in %.

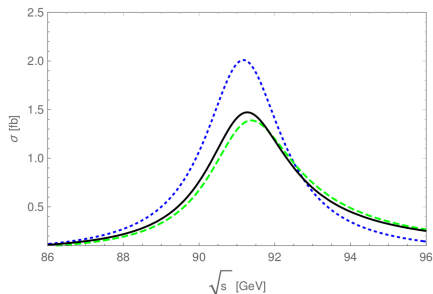
## Numerical Illustration



- Illustration of the Wilson coefficients for the **photonic (I)**, **non-singlet (II)**, **pure-singlet (III)**, **interference  $\times 10$  (IV)** and **neglected contributions  $\times 100$**  multiplied with the factor  $z(1 - z)$ . The black line represents the whole contribution to initial state radiation.



## The Z-Peak



blue: Born

green:  $O(\alpha)$

red:  $O(\alpha^2)$

black:  $O(\alpha^2)$

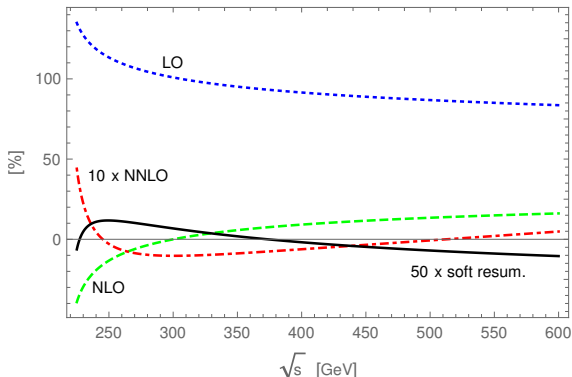
+ soft exponentiation

- ▶ The mass and width of the Z-boson are measured very precisely:  
 $\Delta M_Z = \pm 2.3 \text{ MeV}$ ,  $\Delta \Gamma_Z = \pm 2.1 \text{ MeV}$  (PDG)
- ▶  $O(\alpha^2)$  corrections and soft exponentiation have sizable impact on peak position and width.
- ▶ The differences we found can affect the width of the peak within the accuracy of the experiments.

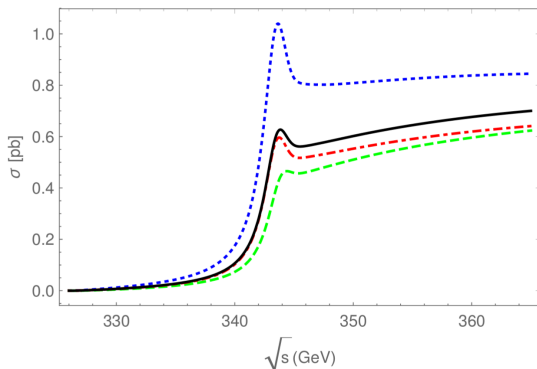
	Fixed width		$s$ dep. width	
	Peak (MeV)	Width (MeV)	Peak (MeV)	Width (MeV)
$O(\alpha)$ correction	210	603	210	602
$O(\alpha^2)$ correction	-109	-187	-109	-187
$O(\alpha^2)$ ; $\gamma$ only	-110	-215	-110	-215
$O(\alpha^2)$ correction				
+ soft exp.	17	23	17	23
Difference to $O(\alpha^2)$ [1]		4		4

$O(34 \text{ MeV})$  shift between fixed and  $s$ -dep. width in peak position

Berends et al., Bardin et al., Beenakker & Hollik.

$Z^0 H^0$  Production

- ▶ Initial-state-radiation has a big effect on the shape of the threshold.
- ▶ The  $\mathcal{O}(\alpha^2)$  corrections are of the size of the anticipated accuracy; the sequence of corrections converges relatively quickly.
- ▶ Soft-photon resummation is less important than in other processes.

$t\bar{t}$  Production at Threshold

- ▶ Initial-state-radiation has a big effect on the shape of the threshold.
- ▶ Soft-photon resummation leads to a sizable effect.
- ▶ The  $\mathcal{O}(\alpha^2)$  corrections are of the size of the anticipated accuracy.

## Opening the Window to even higher orders in $\alpha$ :

The method of massive OMEs allows to calculate all corrections e.g. up to  $O(\alpha^6 L^5)$  [JB, A. De Freitas, K. Schönwald, DESY 19-231]

	$L^6$	$L^5$	$L^4$	$L^3$	$L^2$	$L$	$L^0$
$O(\alpha)$						✓	✓
$O(\alpha^2)$					✓	✓	✓
$O(\alpha^3)$				✓	✓	✓	-
$O(\alpha^4)$			✓	✓	✓	-	-
$O(\alpha^5)$		✓	✓	✓	-	-	-
$O(\alpha^6)$	✓	✓	-	-	-	-	-

## Opening the Window to even higher orders in $\alpha$ :

The method of massive OMEs allows to calculate all corrections e.g. up to  $O(\alpha^6 L^5)$  [JB, A. De Freitas, K. Schönwald, DESY 19-231]

	Fixed width		$s$ dep. width	
	Peak (MeV)	Width (MeV)	Peak (MeV)	Width (MeV)
$O(\alpha)$ correction	185.638	539.408	181.098	524.978
$O(\alpha^2 L^2)$ :	-96.894	-177.147	-95.342	-176.235
$O(\alpha^2 L)$ :	6.982	22.695	6.841	21.896
$O(\alpha^2)$ :	0.176	-2.218	0.174	-2.001
$O(\alpha^3 L^3)$ :	23.265	38.560	22.968	38.081
$O(\alpha^3 L^2)$ :	-1.507	-1.888	-1.491	-1.881
$O(\alpha^3 L)$ :	-0.152	0.105	-0.151	-0.084
$O(\alpha^4 L^4)$ :	-1.857	0.206	-1.858	0.146
$O(\alpha^4 L^3)$ :	0.131	-0.071	0.132	-0.065
$O(\alpha^4 L^2)$ :	0.048	-0.001	0.048	0.001
$O(\alpha^5 L^5)$ :	0.142	-0.218	0.144	-0.212
$O(\alpha^5 L^4)$ :	-0.000	0.020	-0.001	0.020
$O(\alpha^5 L^3)$ :	-0.008	0.009	-0.008	0.008
$O(\alpha^6 L^6)$ :	-0.007	0.027	-0.007	0.027
$O(\alpha^6 L^5)$ :	-0.001	0.000	-0.001	0.000

Table 1: Shifts in the  $Z$ -mass and the width due to the different contributions to the ISR QED radiative corrections for a fixed width of  $\Gamma_Z = 2.4952$  GeV and  $s$ -dependent width using  $M_Z = 91.1876$  GeV and  $s_0 = 4m_\tau^2$ .

- ▶ We calculated the  $O(\alpha^2)$  massive operator matrix elements in QED, which contribute to the 2-loop initial state corrections for  $e^+e^- \rightarrow Z^*/\gamma^*$  in the limit  $m_f^2/s \rightarrow 0$  using the renormalization group method for the electron-contributions.
- ▶ We have obtained all logarithmic contributions  $O((\alpha L)^2)$ ,  $O(\alpha^2 L)$ ,  $O(\alpha L)$  and the constant contributions  $O(\alpha)$  correctly.
- ▶ The **literal** application of the  $s \gg m_f^2$  expansion, as proposed by **BBN seemed for nearly one decade not to deliver** the result obtained by conventional integration. However, we found by a **full calculation** that the  $O(\alpha^2)$  results of BBN are not correct.
- ▶ On the other hand, we obtained our results for the 2-loop matrix elements by two independent methods, which **agree on the results**. Furthermore, the complete OMEs obey Fermion number conservation, and renormalize as expected. The 2-loop anomalous dimensions are correctly obtained.
- ▶ In the case of massless external lines, massive OMEs can be calculated without any problem and the results agree in all cases investigated with that obtained in the limit  $m^2/\mu^2 \rightarrow 0$ . **This now also applies for massive external states.**

- ▶ There are contributions to the  $O(\alpha^2)$  corrections with vanishing OME, which also appear in the massless Drell–Yan process. They have to be included (missing at BBN).
- ▶ Due to the axial-vector couplings of the  $Z$ -boson the corrections in the vector- and axial-vector case are not the same (as already known from the Drell-Yan process). We have accounted for these contributions as well.
- ▶ The differences at  $O(\alpha^2)$  are large, reaching the order of the logarithmic terms in part of the kinematic region.
- ▶ We have performed phenomenological studies for key-processes at present and future  $e^+e^-$  colliders. Cutting at  $s' > 4m_\tau^2$  leads to a difference of **4 MeV** for  $\Gamma_{Z^0}$  scanning the  $Z^0$  peak, at a present accuracy of **2.3 MeV**
- ▶ We have extended the calculation to even higher order ISR corrections:  $O(\alpha^6 L^5)$ .
- ▶ The newly obtained corrections are of relevance at high luminosity  $e^+e^-$  facilities planned for the future.