The 3-Loop Heavy Flavor Corrections to Deep-Inelastic Scattering

Loops and Legs in Quantum Field Theory, Wittenberg, April 14-19, 2024

Johannes Blümlein | April 15, 2024

DESY AND TU DORTMUND

- J. Ablinger et al., The unpolarized and polarized single-mass three-loop heavy flavor operator matrix elements $A_{gg,Q}$ and $\Delta A_{gg,Q}$, JHEP **12** (2022) 134.
- A. Behring, J.B., and K. Schönwald, The inverse Mellin transform via analytic continuation, JHEP **06** (2023) 62.
- J. Ablinger et al., The first-order factorizable contributions to the three-loop massive operator matrix elements $A_{Qg}^{(3)}$ and $\Delta A_{Qg}^{(3)}$, Nucl. Phys.B 999 (2024) 116427.
- J. Ablinger et al., The non-first-order-factorizable contributions to the three-loop single-mass operator matrix elements $A_{Qg}^{(3)}$ and $\Delta A_{Qg}^{(3)}$, 2403.00513 [hep-ph].

The Collaboration

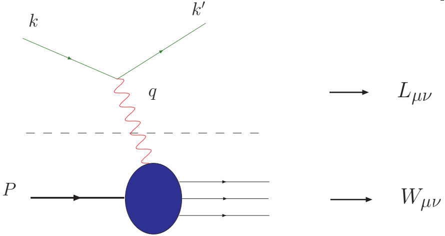
[DESY-JKU Linz & younger colleagues]

- 2007-2009:
  2-loop general N-results and 3-loop moments
  I. Bierenbaum, JB. S. Klein
- 2010-now:
  Individual 3-loop OMEs and HQ Wilson-coefficients at general N and x
  J. Ablinger, A. Behring, JB, A. De Freitas, A. Hasselhuhn, S. Klein, A. von Manteuffel, M. Round, M. Saragnese, C. Schneider, K. Schönwald, F. Wißbrock
- Some special 2-loop applications (including massive QED)
  also: G. Falcioni, W. van Neerven, T. Pfoh, C. Raab

Earlier calculations

- 1976-1982; 1991: Analytic 1-loop results
  E. Witten; J. Babcock, D. W. Sivers, S. Wolfram; M.A. Shifman, A.I. Vainshtein, V.I. Zakharov; J.P. Leveille, T.J. Weiler; M. Glück, E. Hoffmann, E. Reya; C. Watson, W. Vogelsang
- 1995-1998: Analytic 2-loop results
  M. Buza, Y. Matiounine, R. Migneron, W. van Neerven, J. Smith
  1992-1995: Numeric 2-loop results E. Laenen, W. van Neerven, S. Riemersma, J. Smith

Deep–Inelastic Scattering (DIS):



$$Q^2 := -q^2, \quad x := \frac{Q^2}{2P.q} \quad \text{Bjorken–x}$$

$$\frac{d\sigma}{dQ^2\, dx} \sim W_{\mu\nu} L^{\mu\nu}$$

$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi}\int d^4\xi \exp(iq\xi)\langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s\rangle =$$

$$\frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2)$$

$$+ i\varepsilon_{\mu\nu\lambda\sigma}\frac{q^\lambda S^\sigma}{P.q} g_1(x,Q^2) + i\varepsilon_{\mu\nu\lambda\sigma}\frac{q^\lambda(P.qS^\sigma - S.qP^\sigma)}{(P.q)^2} g_2(x,Q^2)\,.$$

The structure functions $F_{2,L}$ and $g_{1,2}$ contain light and heavy quark contributions.
At 3-loop order also graphs with two heavy quarks of different mass contribute.
$\Longrightarrow$ Single and 2-mass contributions: c and b quarks in one graph.

Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x,Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)}_{perturbative} \otimes \underbrace{f_j(x,\mu^2)}_{nonpert.}$$

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs).
$\otimes$ denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz \, \delta(x - yz) f(y) g(z) \, .$$

Many of the subsequent calculations are performed in Mellin space, where $\otimes$ reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) = \int_0^1 dx \, x^{N-1} f(x) \, .$$

Wilson coefficients:

$$\mathbb{C}_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = C_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}\right) + H_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) .$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = \sum_i C_{i,(2,L)}\left(N, \frac{Q^2}{\mu^2}\right) A_{ij}\left(\frac{m^2}{\mu^2}, N\right)$$

[Buza, Matiounine, Smith, van Neerven 1996]
factorizes into the light flavor Wilson coefficients C and the massive operator matrix elements (OMEs) of local operators O_i between partonic states j

$$A_{ij}\left(\frac{m^2}{\mu^2}, N\right) = \langle j \mid O_i \mid j \rangle .$$

$\rightarrow$ additional Feynman rules with local operator insertions for partonic matrix elements.
The unpolarized light flavor Wilson coefficients are known up to NNLO
[Vermaseren, Moch, Vogt, 2005; JB, Marquard, Schneider, Schönwald, 2022].
For $F_2(x, Q^2)$: at $Q^2 \gtrsim 10 m^2$ the asymptotic representation holds at the 1% level.

The main time-line for the 3-loop corrections

- 2005 F_L [no massive 3-loop OMEs needed]
- 2010 All unpolarized N_F terms and $A_{qg,Q}^{(3)}, A_{qq,Q}^{(3).PS}$
- 2014 unpolarized logarithmic 3-loop contributions and $A_{gq,Q}^{(3)}, (\Delta)A_{qq,Q}^{(3),NS}, A_{Qq}^{(3),PS}$
- 2017 two-mass corrections $A_{gq,Q}^{(3)}, (\Delta)A_{qq,Q}^{(3),NS}, A_{Qq}^{(3),PS}$
- 2018 two-mass corrections $A_{gg,Q}^{(3)}$
- 2019 2-loop correction: $(\Delta)A_{Qq}^{(2),PS}$ whole kinematic region and $\Delta A_{Qq}^{(3),PS}$
- 2019 two-mass corrections $\Delta A_{Qq}^{(3),PS}$
- 2020 two-mass corrections $\Delta A_{gg,Q}^{(3)}$
- 2021 polarized logarithmic 3-loop contributions and $\Delta A_{gq,Q}^{(3)}, \Delta A_{qq,Q}^{(3).PS}, \Delta A_{gq}^{(3)}$
- 2022 3-loop polarized massless Wilson coefficients [JB, Marquard, Schneider, Schönwald]
- 2022 corrected the polarized 2-loop contributions
- 2022 $(\Delta)A_{gg,Q}^{(3)}$
- 2023 $(\Delta)A_{Qg}^{(3)}$: 1st order factorizing parts
- 2024 $(\Delta)A_{Qg}^{(3)}$, [two-mass corrections $(\Delta)A_{Qg}^{(3)}$]

- 45 physics papers (journals)
- 26 mathematical papers
  - 1998 Harmonic sums [Vermaseren; JB]
  - 2000,2005 Analytic continuations of harmonic sums to $N \in \mathbb{C}$ [JB; JB, S. Moch]
  - 2003 Concrete shuffle algebras [JB]
  - 2009 Guessing large recurrences [JB, M. Kauers, S. Klein, C. Schneider]
  - 2009 Structural relations of harmonic sums [JB]
  - 2009 MZV Data mine [JB, D. Broadhurst, J. Vermaseren]
  - 2011 Cyclotomic harmonic sums and harmonic polylogarithms [Ablinger, JB, Schneider]
  - 2013 Generalized harmonic sums and harmonic polylogarithms [Ablinger, JB, Schneider]; 2001 [Moch, Uwer, Weinzierl]
  - 2014 Finite binomial sums and root-valued iterated integrals [Ablinger, JB, Raab, Schneider]
  - 2017 ${}_2F_1$ solutions (iterated non-iterative integrals) [J. Ablinger, JB, A. De Freitas, M. van Hoeij, E. Imamoglu, C. Raab, C.S. Radu, C. Schneider]
  - 2017 Methods of arbitrary high moments [JB, Schneider]
  - 2018 Automated solution of first-order factorizing differential equation systems in an arbitrary basis [J. Ablinger, JB, P. Marquard, N. Rana, C. Schneider]
  - 2023 Analytic continuation form t to x-space [JB, Behring, Schönwald]

Important Computer-Algebra Packages

C. Schneider: `Sigma, EvaluateMultiSums, SumProduction, SolveCoupledSystem`
J. Ablinger: `HarmonicSums`

The Wilson Coefficients at large Q^2

$$L_{q,(2,L)}^{\rm NS}(N_F+1) = a_s^2\left[A_{qq,Q}^{(2),\rm NS}(N_F+1)\delta_2 + \hat{C}_{q,(2,L)}^{(2),\rm NS}(N_F)\right] + a_s^3\left[A_{qq,Q}^{(3),\rm NS}(N_F+1)\delta_2 + A_{qq,Q}^{(2),\rm NS}(N_F+1)C_{q,(2,L)}^{(1),\rm NS}(N_F+1) + \hat{C}_{q,(2,L)}^{(3),\rm NS}(N_F)\right]$$

$$L_{q,(2,L)}^{\rm PS}(N_F+1) = a_s^3\left[A_{qq,Q}^{(3),\rm PS}(N_F+1)\delta_2 + N_F A_{gq,Q}^{(2),\rm NS}(N_F)\tilde{C}_{g,(2,L)}^{(1),\rm NS}(N_F+1) + N_F \hat{\tilde{C}}_{q,(2,L)}^{(3),\rm PS}(N_F)\right]$$

$$\begin{aligned} L_{g,(2,L)}^{\rm S}(N_F+1) = {} & a_s^2 A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1) + a_s^3\Big[A_{qg,Q}^{(3)}(N_F+1)\delta_2 + A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1) \\ & + A_{gg,Q}^{(2)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + A_{Qg}^{(1)}(N_F+1)N_F\tilde{C}_{q,(2,L)}^{(2),\rm PS}(N_F+1) + N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\Big] \end{aligned}$$

$$\begin{aligned} H_{q,(2,L)}^{\rm PS}(N_F+1) = {} & a_s^2\left[A_{Qq}^{(2),\rm PS}(N_F+1)\delta_2 + \tilde{C}_{q,(2,L)}^{(2),\rm PS}(N_F+1)\right] \\ & + a_s^3\Big[A_{Qq}^{(3),\rm PS}(N_F+1)\delta_2 + A_{gq,Q}^{(2)}(N_F+1)\tilde{C}_{g,(1,L)}^{(2)}(N_F+1) + A_{Qq}^{(2),\rm PS}(N_F+1)\tilde{C}_{q,(2,L)}^{(1),\rm NS}(N_F+1) + \tilde{C}_{q,(2,L)}^{(3),\rm PS}(N_F+1)\Big] \end{aligned}$$

$$\begin{aligned} H_{g,(2,L)}^{\rm S}(N_F+1) = {} & a_s\left[A_{Qg}^{(1)}(N_F+1)\delta_2 + \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\right] \\ & + a_s^2\Big[A_{Qg}^{(2)}(N_F+1)\delta_2 + A_{Qg}^{(1)}(N_F+1)\tilde{C}_{q,(2,L)}^{(1)}(N_F+1) + A_{gg,Q}^{(1)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\Big] \\ & + a_s^3\Big[A_{Qg}^{(3)}(N_F+1)\delta_2 + A_{Qg}^{(2)}(N_F+1)\tilde{C}_{q,(2,L)}^{(1)}(N_F+1) + A_{gg,Q}^{(2)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\ & + A_{Qg}^{(1)}(N_F+1)\tilde{C}_{q,(2,L)}^{(2),\rm S}(N_F+1) + A_{gg,Q}^{(1)}(N_F+1)\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \tilde{C}_{g,(2,L)}^{(3)}(N_F+1)\Big] \end{aligned}$$

- The case for two different masses obeys an analogous representation.
- Note the contributions of the massless Wilson coefficients.

The variable flavor number scheme

- Matching conditions for parton distribution functions:

$$f_k(N_F+2)+f_{\overline{k}}(N_F+2) = A_{qq,Q}^{\rm NS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot\left[f_k(N_F)+f_{\overline{k}}(N_F)\right]+\frac{1}{N_F}A_{qq,Q}^{\rm PS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot\Sigma(N_F)$$
$$+\frac{1}{N_F}A_{qg,Q}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot G(N_F)\,,$$

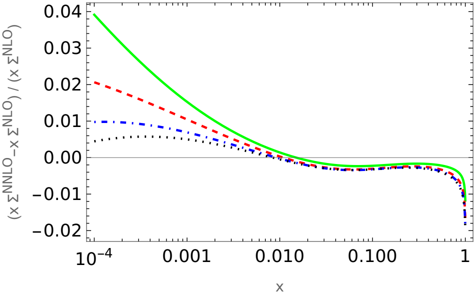
$$f_Q(N_F+2)+f_{\overline{Q}}(N_F+2) = A_{Qq}^{\rm PS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2},\right)\cdot\Sigma(N_F)+A_{Qg}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot G(N_F)\,,$$

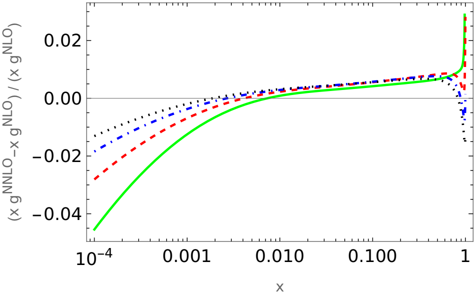
$$\Sigma(N_F+2) = \left[A_{qq,Q}^{\rm NS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)+A_{qq,Q}^{\rm PS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)+A_{Qq}^{\rm PS}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\right]\cdot\Sigma(N_F)$$
$$+\left[A_{qg,Q}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)+A_{Qg}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\right]\cdot G(N_F)\,,$$

$$G(N_F+2) = A_{gq,Q}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot\Sigma(N_F)+A_{gg,Q}\left(N_F+2,\frac{m_1^2}{\mu^2},\frac{m_2^2}{\mu^2}\right)\cdot G(N_F)\,.$$

The charm and bottom quark masses are not that much different.

Relative effect in unpolarized NNLO evolution

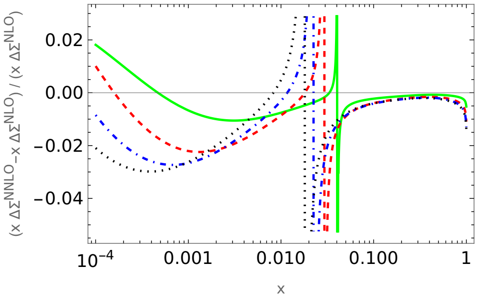


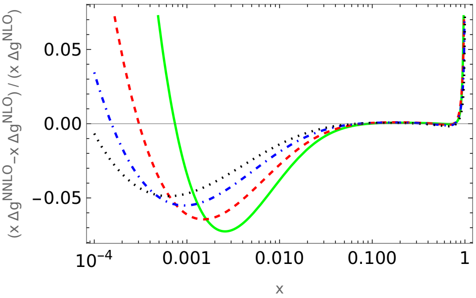


$Q^2 = 10, 10^2, 10^3, 10^4$ GeV^2 dotted, dash-dotted, dashed, full lines. [M. Saragnese, 2022]

The unpolarized world deep-inelastic data have a precision of $O(1\%)$.

Relative effect in polarized NNLO evolution



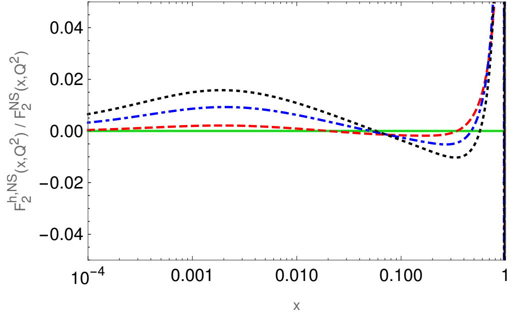


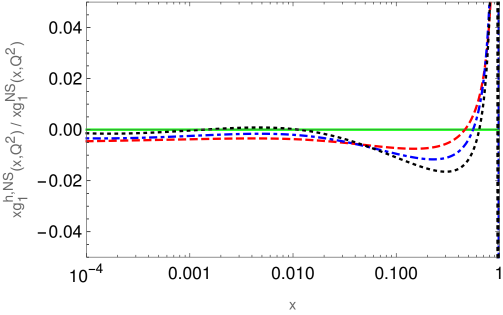
$Q^2 = 10, 10^2, 10^3, 10^4$ GeV^2 dotted, dash-dotted, dashed, full lines. [M. Saragnese, 2022]

The future polarized data at the EIC will reach a precision of $O(1\%)$.

The relative contribution of HQ to non-singlet structure functions at N^{3}LO

Scheme-invariant evolution

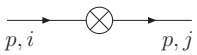




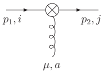
Left: The relative contribution of the heavy flavor contributions due to *c* and *b* quarks to the structure function $F_2^{\rm NS}$ at N^{3}LO; dashed lines: 100 GeV^2; dashed-dotted lines: 1000 GeV^2; dotted lines: 10000 GeV^2. Right: The same for the structure function $xg_1^{\rm NS}$ at N^{3}LO. [JB, M. Saragnese, 2021].

Calculation of the 3-loop operator matrix elements

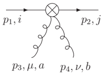
The OMEs are calculated using the QCD Feynman rules together with the following operator insertion Feynman rules:



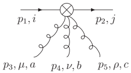
$$\delta^{ij} \not{\Delta} \gamma_\pm (\Delta \cdot p)^{N-1} , \quad N \geq 1$$



$$g t^a_{ji} \Delta^\mu \not{\Delta} \gamma_\pm \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2} , \quad N \geq 2$$



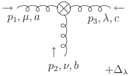
$$g^2 \Delta^\mu \Delta^\nu \not{\Delta} \gamma_\pm \sum_{j=0}^{N-3} \sum_{l=j+1}^{N-2} (\Delta p_2)^j (\Delta p_1)^{N-l-2}$$
$$\left[(t^a t^b)_{ji} (\Delta p_1 + \Delta p_4)^{l-j-1} + (t^b t^a)_{ji} (\Delta p_1 + \Delta p_3)^{l-j-1} \right] ,$$
$$N \geq 3$$



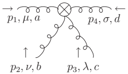
$$g^3 \Delta_\mu \Delta_\nu \Delta_\rho \not{\Delta} \gamma_\pm \sum_{j=0}^{N-4} \sum_{l=j+1}^{N-3} \sum_{m=l+1}^{N-2} (\Delta . p_2)^j (\Delta . p_1)^{N-m-2}$$
$$\Big[\ (t^a t^b t^c)_{ji} (\Delta . p_4 + \Delta . p_5 + \Delta . p_1)^{l-j-1} (\Delta . p_5 + \Delta . p_1)^{m-l-1}$$
$$+ (t^a t^c t^b)_{ji} (\Delta . p_4 + \Delta . p_5 + \Delta . p_1)^{l-j-1} (\Delta . p_4 + \Delta . p_1)^{m-l-1}$$
$$+ (t^b t^a t^c)_{ji} (\Delta . p_3 + \Delta . p_5 + \Delta . p_1)^{l-j-1} (\Delta . p_5 + \Delta . p_1)^{m-l-1}$$
$$+ (t^b t^c t^a)_{ji} (\Delta . p_3 + \Delta . p_5 + \Delta . p_1)^{l-j-1} (\Delta . p_3 + \Delta . p_1)^{m-l-1}$$
$$+ (t^c t^a t^b)_{ji} (\Delta . p_3 + \Delta . p_4 + \Delta . p_1)^{l-j-1} (\Delta . p_4 + \Delta . p_1)^{m-l-1}$$
$$+ (t^c t^b t^a)_{ji} (\Delta . p_3 + \Delta . p_4 + \Delta . p_1)^{l-j-1} (\Delta . p_3 + \Delta . p_1)^{m-l-1} \Big] ,$$
$$N \geq 4$$

$$\gamma_+ = 1 , \quad \gamma_- = \gamma_5 .$$

$$\frac{1+(-1)^N}{2} \delta^{ab} (\Delta \cdot p)^{N-2}$$
$$\left[g_{\mu\nu} (\Delta \cdot p)^2 - (\Delta_\mu p_\nu + \Delta_\nu p_\mu) \Delta \cdot p + p^2 \Delta_\mu \Delta_\nu \right] , \quad N \geq 2$$



$$-ig \frac{1+(-1)^N}{2} f^{abc} \Bigg($$
$$\left[(\Delta_\nu g_{\lambda\mu} - \Delta_\lambda g_{\mu\nu}) \Delta \cdot p_1 + \Delta_\mu (p_{1,\nu} \Delta_\lambda - p_{1,\lambda} \Delta_\nu) \right] (\Delta \cdot p_1)^{N-2}$$
$$+ \Delta_\lambda \left[\Delta \cdot p_1 p_{2,\mu} \Delta_\nu + \Delta \cdot p_2 p_{1,\nu} \Delta_\mu - \Delta \cdot p_1 \Delta \cdot p_2 g_{\mu\nu} - p_1 \cdot p_2 \Delta_\mu \Delta_\nu \right]$$
$$\times \sum_{j=0}^{N-3} (-\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-3-j}$$
$$+ \left\{ \begin{smallmatrix} p_1 \to p_2 \to p_3 \to p_1 \\ \mu \to \nu \to \lambda \to \mu \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} p_1 \to p_3 \to p_2 \to p_1 \\ \mu \to \lambda \to \nu \to \mu \end{smallmatrix} \right\} \Bigg) , \quad N \geq 2$$



$$g^2 \frac{1+(-1)^N}{2} \Bigg(f^{abe} f^{cde} O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4)$$
$$+ f^{ace} f^{bde} O_{\mu\lambda\nu\sigma}(p_1, p_3, p_2, p_4) + f^{ade} f^{bce} O_{\mu\sigma\nu\lambda}(p_1, p_4, p_2, p_3) \Bigg) ,$$
$$O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4) = \Delta_\nu \Delta_\lambda \Big\{ -g_{\mu\sigma} (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-2}$$
$$+ [p_{4,\mu} \Delta_\sigma - \Delta \cdot p_4 g_{\mu\sigma}] \sum_{i=0}^{N-3} (\Delta \cdot p_3 + \Delta \cdot p_4)^i (\Delta \cdot p_4)^{N-3-i}$$
$$- [p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 g_{\mu\sigma}] \sum_{i=0}^{N-3} (-\Delta \cdot p_1)^i (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-3-i}$$
$$+ [\Delta \cdot p_1 \Delta \cdot p_4 g_{\mu\sigma} + p_1 \cdot p_4 \Delta_\mu \Delta_\sigma - \Delta \cdot p_4 p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 p_{4,\mu} \Delta_\sigma]$$
$$\times \sum_{i=0}^{N-4} \sum_{j=0}^{i} (-\Delta \cdot p_1)^{N-4-i} (\Delta \cdot p_3 + \Delta \cdot p_4)^{i-j} (\Delta \cdot p_4)^j \Big\}$$
$$- \left\{ \begin{smallmatrix} p_1 \leftrightarrow p_2 \\ \mu \leftrightarrow \nu \end{smallmatrix} \right\} - \left\{ \begin{smallmatrix} p_3 \leftrightarrow p_4 \\ \lambda \leftrightarrow \sigma \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} p_1 \leftrightarrow p_2, \ p_3 \leftrightarrow p_4 \\ \mu \leftrightarrow \nu, \ \lambda \leftrightarrow \sigma \end{smallmatrix} \right\} , \quad N \geq 2$$

Calculation methods

- Diagram generation: `QGRAF` [Nogueira, 1993]
- Lorentz and Dirac algebra: `Form` [Vermaseren, 2000]
- Color algebra: `Color` [van Ritbergen, Schellekens, Vermaseren, 1999]
- IBP reduction: `Reduze 2` [von Manteuffel, Studerus 2009,2012]
- *N* space calculations:
  - Method of arbitrary large moments [JB, Schneider, 2017]
  - Summation theory and solving first-order factorizing recurrences: `Sigma` [Schneider, 2007,2013]
  - Reduce the results in the respective function spaces: `HarmonicSums` [Ablinger, 2009, 2012, etc.]
- *x* space calculations
  - solve 1st order factorizing differential equations
  - transform from $N \to t$-space, solve the respective systems of differential equations (not necessarily factorizing to first order) [Behring, JB, Schönwald, 2023]
  - Reduce the results in the respective function spaces; iterated integrals over alphabets containing also higher transcendental letters [Ablinger et al. 2017]
  - The higher transcendental letters have to be known in analytic form for $z \in \mathbb{C}$.
- Both *N* and *x* space techniques are needed to solve the present problem. The recurrences for $A_{Qg}^{(3)}$ need far more than 15000 moments to be found & there are no technologies yet to solve non-first order factorizing recurrences analytically.
- Final numerical representation: In the most complicated cases: local series expansions in *x* at high precision.

Mathematical Background

- massless and massive contributions to two-loops: harmonic sums
- all pole terms to three-loops: harmonic sums
- all massless Wilson coefficients to three-loops: harmonic sums

Single-mass OMEs

- all N_F of the massive OMEs three-loops: harmonic sums
- $(\Delta)A_{qq,Q}^{(3),\mathrm{NS}}, (\Delta)A_{gq,Q}^{(3)}, (\Delta)A_{qg,Q}^{(3)}, (\Delta)A_{qq,Q}^{(3),\mathrm{PS}}$ to three-loops: harmonic sums
- $(\Delta)A_{Qq}^{(3),\mathrm{PS}}$ to three-loops: generalized harmonic sums and also $H_{\vec{a}}(1-2x)$
- $(\Delta)A_{gg,Q}^{(3)}$ to three-loops: finite binomial sums and square-root valued iterated integrals
- $(\Delta)A_{Qg}^{(3)}$ to three-loops:
  - first-order factorizing contributions: finite binomial sums; all iterated integrals in x-space can be rationalized
  - non-first-order factorizing contributions: ${}_2F_1$ letters in iterated integrals in x-space

Two-mass OMEs

- $(\Delta)A_{qq,Q}^{(3),\mathrm{NS}}, (\Delta)A_{gq,Q}^{(3)}$: harmonic sums
- $(\Delta)A_{Qq}^{(3),PS}$: analytic solutions in x-space only; different supports; root-values letters
- $(\Delta)A_{gg,Q}^{(3)}$: root-valued iterated integrals

Inverse Mellin transform via analytic continuation: $a_{Qg}^{(3)}$

Resumming Mellin N into a continuous variable t, observing crossing relations. Ablinger et al. 2012

$$\sum_{k=0}^{\infty} t^k (\Delta.p)^k \frac{1}{2}[1 \pm (-1)^k] = \frac{1}{2}\left[\frac{1}{1 - t\Delta.p} \pm \frac{1}{1 + t\Delta.p}\right]$$

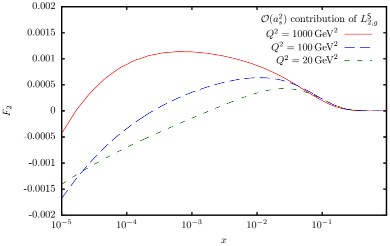
$$\mathfrak{A} = \{f_1(t), ..., f_m(t)\}, \qquad \mathrm{G}(b, \vec{a}; t) = \int_0^t dx_1 f_b(x_1) \mathrm{G}(\vec{a}; x_1), \qquad \left[\frac{d}{dt}\frac{1}{f_{a_{k-1}}(t)}\frac{d}{dt} \cdots \frac{1}{f_{a_1}(t)}\frac{d}{dt}\right] \mathrm{G}(\vec{a}; t) = f_{a_k}(t).$$

The $f_i(t)$ include higher transcendental letters. Regularization for $t \to 0$ needed.

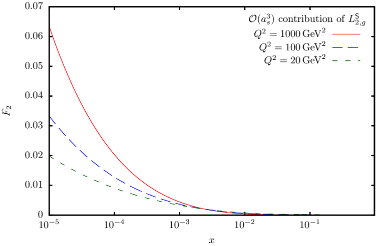
$$\begin{aligned} F(N) &= \int_0^1 dx x^{N-1}[f(x) + (-1)^{N-1} g(x)] \\ \tilde{F}(t) &= \sum_{N=1}^{\infty} t^N F(N) \\ f(x) + (-1)^{N-1} g(x) &= \frac{1}{2\pi i}\left[\mathrm{Disc}_x \tilde{F}\left(\frac{1}{x}\right) + (-1)^{N-1}\mathrm{Disc}_x \tilde{F}\left(-\frac{1}{x}\right)\right]. \end{aligned} \qquad (1)$$

t-space is still Mellin space. One needs closed expressions to perform the analytic continuation (1). Analytic continuation is needed to calculate the small x behaviour. The final expansion maps the problem into a very large number of G-constants, including those with higher transcendental letters.

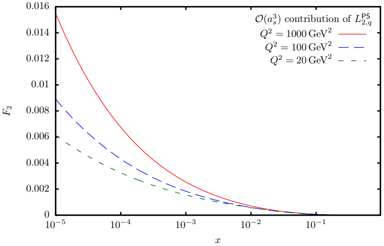
Numerical Results : $L_{g,2}^{\mathrm{S}}$ and $L_{q,2}^{\mathrm{PS}}$



$O(a_s^2)$ $L_{2,g}^{\mathrm{S}}$

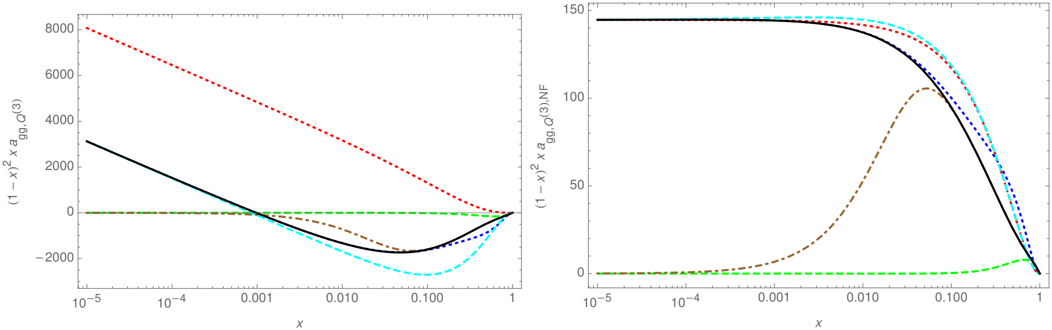


$O(a_s^3)$ $L_{2,g}^{\mathrm{S}}$



$L_{q,2}^{\mathrm{PS}}$

$a_{gg}^{(3)}$

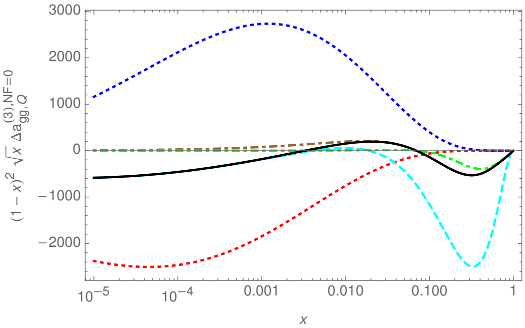


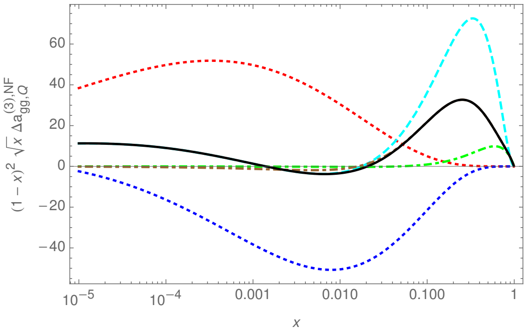
non N_F terms

N_F terms

Left panel: The non–N_F terms of $a_{gg,Q}^{(3)}(N)$ (rescaled) as a function of x. Full line (black): complete result; upper dotted line (red): term $\propto \ln(x)/x$; lower dashed line (cyan): small x terms $\propto 1/x$; lower dotted line (blue): small x terms including all $\ln(x)$ terms up to the constant term; upper dashed line (green): large x contribution up to the constant term; dash-dotted line (brown): complete large x contribution. Right panel: the same for the N_F contribution.

$\Delta a_{gg}^{(3)}$



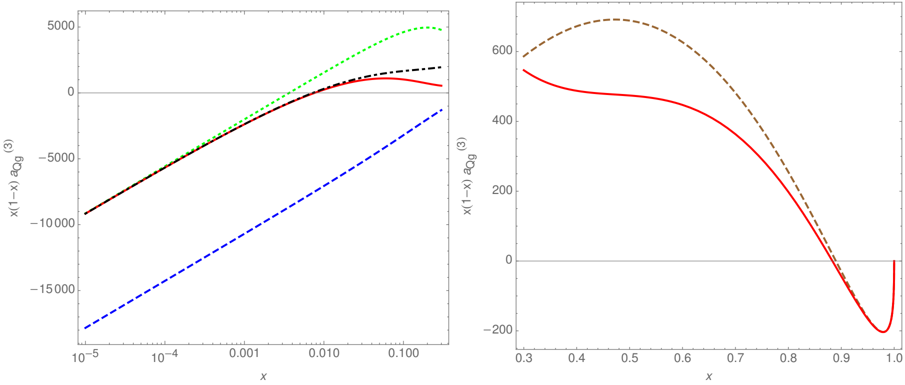


non N_F terms $\qquad$ N_F terms

The non–N_F terms of $\Delta a_{gg,Q}^{(3)}(N)$ (rescaled) as a function of x. Full line (black): complete result; lower dotted line (red): term $\ln^5(x)$; upper dotted line (blue): small x terms $\propto \ln^5(x)$ and $\ln^4(x)$; upper dashed line (cyan): small x terms including all $\ln(x)$ terms up to the constant term; lower dash-dotted line (green): large x contribution up to the constant term; dash-dotted line (brown): full large x contribution. Right panel: the same for the N_F contribution.

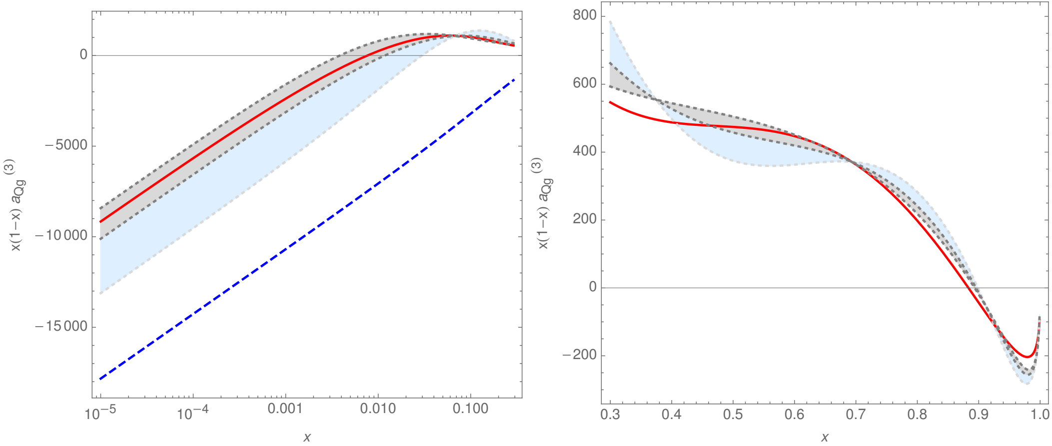
$a_{Qg}^{(3)}$

1009 of the total 1233 Feynman diagrams have first-order factorizing contributions only and are given by G-functions up to root-values letters. The letters for all constants can be rationalized.



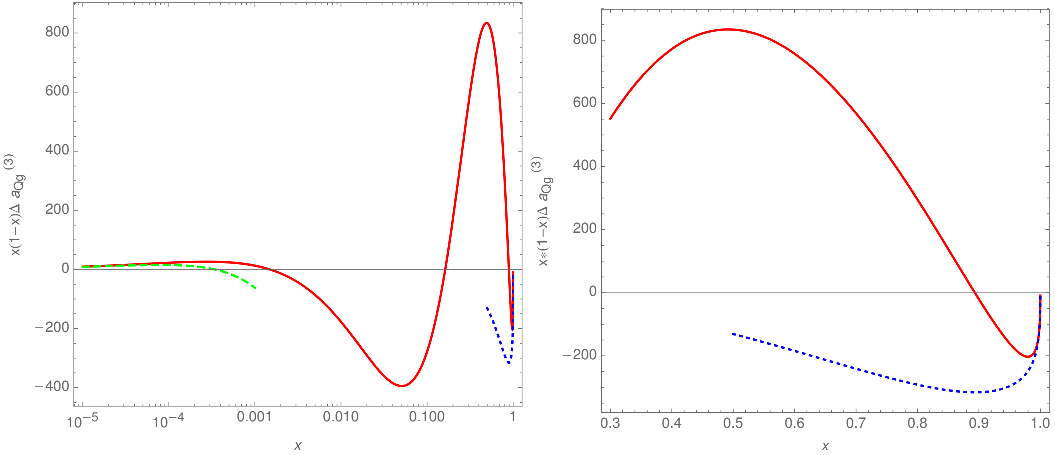
$a_{Qg}^{(3)}(x)$ as a function of x, rescaled by the factor $x(1-x)$. Left panel: smaller x region. Full line (red): $a_{Qg}^{(3)}(x)$; dashed line (blue): leading small-x term $\propto \ln(x)/x$ [Catani, Ciafaloni, Hautmann, 1990]; dotted line (green): $\ln(x)/x$ and $1/x$ term; dash-dotted line (black): all small-x terms, including also $\ln^k(x)$, $k \in \{1, \dots, 5\}$. Right panel: larger x region. Full line (red): $a_{Qg}^{(3)}(x)$; dashed line (brown): leading large-x terms up to the terms $\propto (1-x)$, covering the logarithmic contributions of $O(\ln^k(1-x))$, $k \in \{1, 4\}$.

$a_{Qg}^{(3)}$



$a_{Qg}^{(3)}(x)$ as a function of x, rescaled by the factor $x(1-x)$. Left panel: smaller x region. Full line (red): $a_{Qg}^{(3)}(x)$; dashed line (blue): leading small-x term $\propto \ln(x)/x$ [Catani, Ciafaloni, Hautmann, 1990]; light blue region: estimates of [Kawamura et al., 2012]; gray region: estimates of [ABMP 2017]. Right panel: larger x region. Full line (red): $a_{Qg}^{(3)}(x)$; light blue region: estimates of [Kawamura et al., 2012] gray region: estimates of [ABMP 2017].

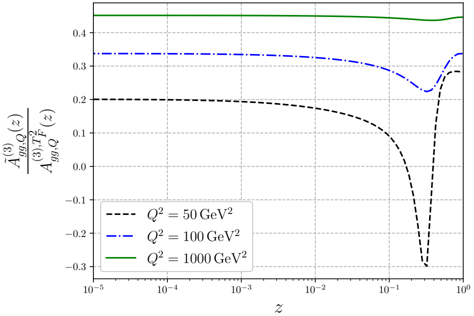
$\Delta a_{Qg}^{(3)}$



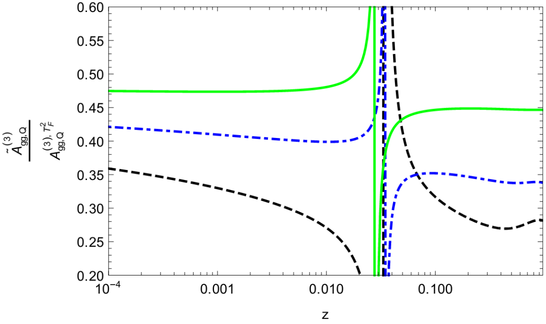
$\Delta a_{Qg}^{(3)}(x)$ as a function of x, rescaled by the factor $x(1-x)$. Left panel: full line (red): $\Delta a_{Qg}^{(3)}(x)$; dashed line (green): the small-x terms $\ln^k(x)$, $k \in \{1, \ldots, 5\}$; dotted line (blue): the large-x terms $\ln^l(1-x)$, $l \in \{1, \ldots, 4\}$. Right panel: larger x region. Full line (red): $\Delta a_{Qg}^{(3)}(x)$; dotted line (blue): the large-x terms $\ln^l(1-x)$, $l \in \{1, \ldots, 4\}$.

Two-mass Results: $\tilde{A}_{gg,Q}^{(3)}$

The two mass contributions over the whole T_F^2-contributions to the OME $\tilde{A}_{gg,Q}^{(3)}$:

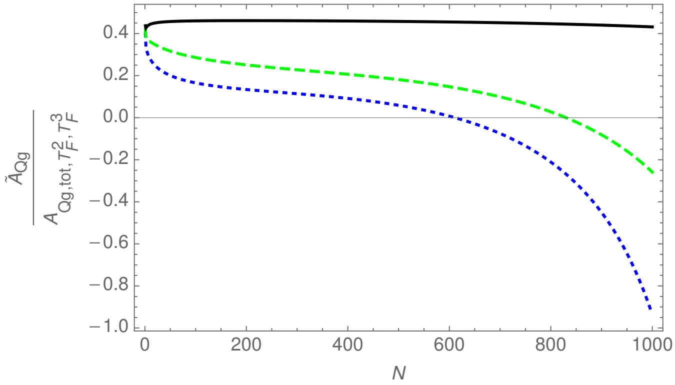


unpolarized



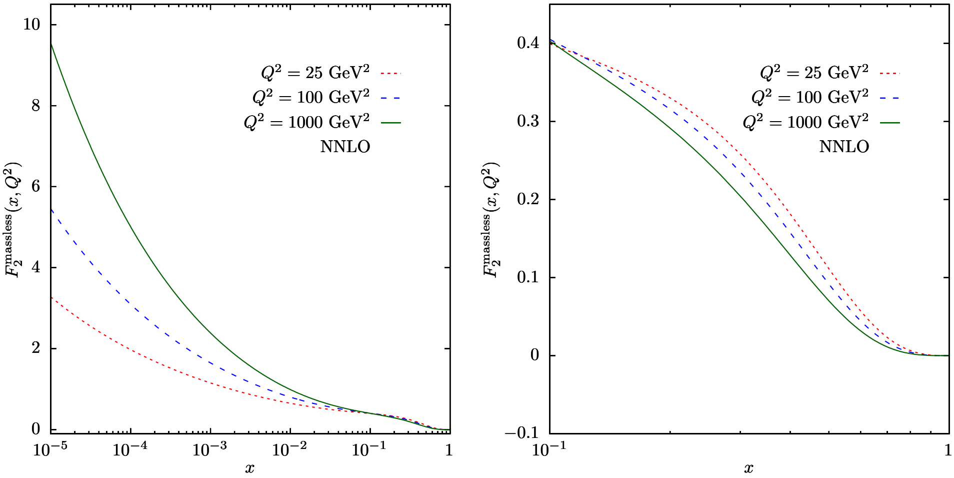
polarized

Relative contribution of $\tilde{A}_{Qg}^{(3)}(N)$



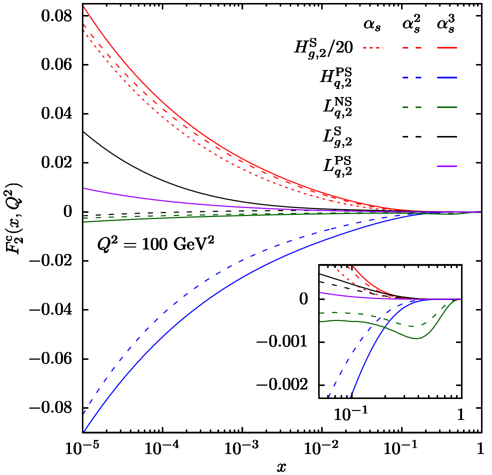
$Q^2 = 30$ GeV^2: dotted line; $Q^2 = 10^2$ GeV^2: dashed line; $Q^2 = 10^4$ GeV^2: full line.

The massless contributions to F_2

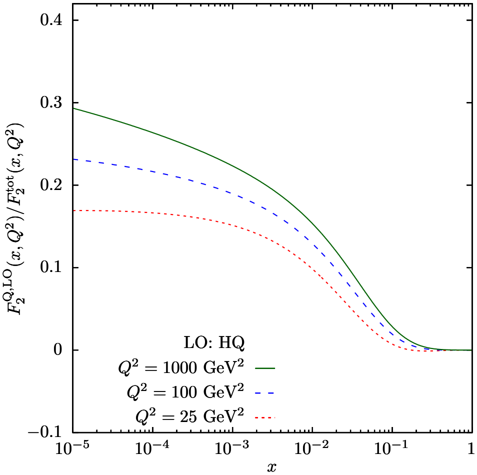


$N_F = 3$ massless quarks.

Single-mass contributions to $F_2^{c,b}$



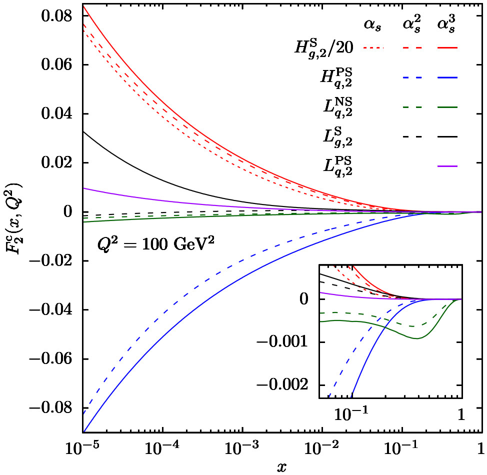
charm for $Q^2 = 100\ \mathrm{GeV}^2$.



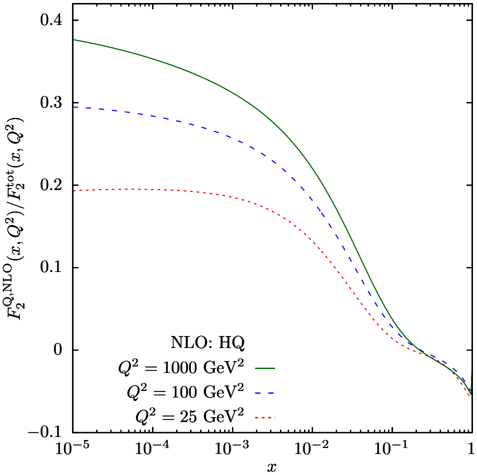
c and *b* single mass contributions

Allows to strongly reduce the current theory error on m_c.

Single-mass contributions to $F_2^{c,b}$



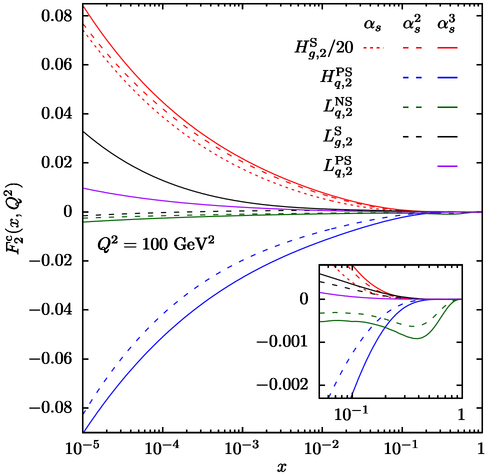
charm for $Q^2 = 100$ GeV^2.

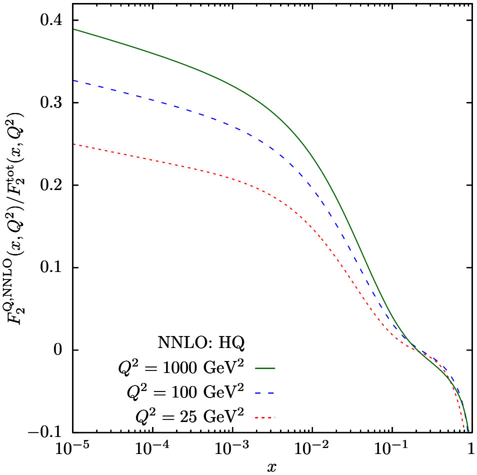


c and *b* single mass contributions

Allows to strongly reduce the current theory error on m_c.

Single-mass contributions to $F_2^{c,b}$





charm for $Q^2 = 100$ GeV^2. *c* and *b* single mass contributions

Allows to strongly reduce the current theory error on m_c.

Conclusions

- All unpolarized and polarized single-mass OMEs and the associated massive Wilson coefficients for $Q^2 \gg m_Q^2$ have been calculated. The unpolarized and polarized massless three-loop Wilson coefficients were calculated and contribute to the present results.
- The calculation of all unpolarized and polarized two-mass OMEs, except for $(\Delta)A_{Qg}^{(3)}$, are finished and the remaining OMEs will be available very soon.
- Various new mathematical and technological methods were developed during the present project. They are available for use in further single- and two-mass calculations in other QFT projects.
- Very soon new precision analyses of the world DIS-data to measure $\alpha_s(M_Z)$ and m_c at higher precision can be carried out.
- Both the single- and two-mass VFNS at 3-loop order will be available in form of a numerical program, to be used e.g. in applications at hadron colliders.
- The results in the polarized case prepare the analysis of the precision data, which will be taken at the EIC starting at the end of this decade.
- For all sub-processes it turned out that the small x BFKL approaches fail to present the physical result due to quite a series of missing subleading terms, which substantially correct the LO behaviour. The correct description requires the full calculation.