3-Loop Corrections to Heavy Flavor Wilson Coefficients in Deep-Inelastic Scattering

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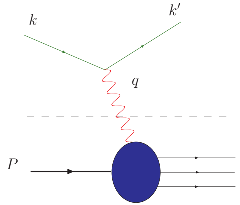
Introduction

Unpolarized Deep–Inelastic Scattering (DIS):



$\longrightarrow \quad L_{\mu\nu}$

$\longrightarrow \quad W_{\mu\nu}$

$$Q^2 := -q^2, \quad x := \frac{Q^2}{2P.q} \quad \text{Bjorken–x}$$

$$\frac{d\sigma}{dQ^2\, dx} \sim W_{\mu\nu} L^{\mu\nu}$$

$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi}\int d^4\xi \exp(iq\xi)\langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s\rangle =$$

$$\frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2) \,.$$

Structure Functions: $F_{2,L}$
contain light and heavy quark contributions.

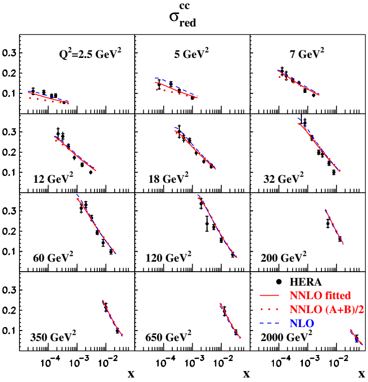
$\alpha_s(M_Z^2)$ from NNLO DIS(+) analyses

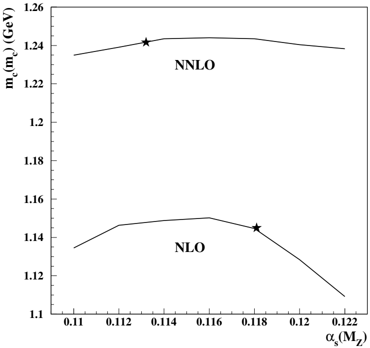
	$\alpha_s(M_Z^2)$	
BBG	$0.1134\ ^{+0.0019}_{-0.0021}$	valence analysis, NNLO
GRS	0.112	valence analysis, NNLO
ABKM	0.1135 ± 0.0014	HQ: FFNS $N_f = 3$
JR	0.1128 ± 0.0010	dynamical approach
JR	0.1162 ± 0.0006	including NLO-jets
MSTW	0.1171 ± 0.0014	
Thorne	0.1136	[DIS+DY+HT*] (2014)
$\mathrm{ABM11}_J$	$0.1134 - 0.1149 \pm 0.0012$	Tevatron jets (NLO) incl.
ABM13	0.1133 ± 0.0011	
ABM13	0.1132 ± 0.0011	(without jets)
ABM16	0.1149 ± 0.0009	+ new HERA, + $t\bar{t}$
CTEQ	0.1159..0.1162	
CTEQ	0.1140	(without jets)
NN21	$0.1174 \pm 0.0006 \pm 0.0001$	
Gehrmann et al.	$0.1131\ ^{+\,0.0028}_{-\,0.0022}$	e^+e^- thrust
Abbate et al.	0.1140 ± 0.0015	e^+e^- thrust
BBG	$0.1141\ ^{+0.0020}_{-0.0022}$	valence analysis, $\mathrm{N^3LO}$

$$\Delta_{\rm TH}\alpha_s = \alpha_s(\mathrm{N^3LO}) - \alpha_s(\mathrm{NNLO}) + \Delta_{\rm HQ} = +0.0009 \pm 0.0006_{\rm HQ}$$

NNLO accuracy is needed to analyze the world data. $\Longrightarrow$ NNLO HQ corrections needed.

Deep–Inelastic Scattering (DIS):





NNLO:

S. Alekhin, J. Blümlein, K. Daum, K. Lipka, Phys.Lett. B720 (2013) 172 [1212.2355]

$$m_c(m_c) = 1.25 \pm 0.02(exp) \begin{array}{c} +0.03 \\ -0.02 \end{array} (scale) \begin{array}{c} +0.00 \\ -0.07 \end{array} (thy),$$

$$m_b(m_b) = 3.91 \pm 0.14(exp) \begin{array}{c} +0.00 \\ -0.11 \end{array} (thy) \text{ (preliminary)},$$

$$m_t(m_t) = 160.9 \pm 1.2(exp) \text{ GeV (preliminary)}$$

Yet approximate NNLO treatment [Kawamura et al. [1205.5227]].

Publications: Physics

JB, A. De Freitas, S. Klein, W.L. van Neerven, Nucl. Phys. B755 (2006) 272
I. Bierenbaum, JB, S. Klein, Nucl. Phys. B780 (2007) 40; Nucl.Phys. B820 (2009) 417; Phys.Lett. B672 (2009) 401
JB, S. Klein, B. Tödtli, Phys. Rev. D80 (2009) 094010
I. Bierenbaum, JB, S. Klein, C. Schneider, Nucl. Phys. B803 (2008) 1
J. Ablinger, JB, S. Klein, C. Schneider, F. Wißbrock, Nucl. Phys. B844 (2011) 26
JB, A. Hasselhuhn, S. Klein, C. Schneider, Nucl. Phys. B866 (2013) 196
J. Ablinger et al., Nucl. Phys. B864 (2012) 52; Nucl. Phys. B882 (2014) 263; Nucl. Phys. B885 (2014) 409; Nucl. Phys. B885 (2014) 280; Nucl. Phys. B886 (2014) 733; Nucl.Phys. B890 (2014) 48; DESY 15-049.
A. Behring et al., Eur.Phys.J. C74 (2014) 9, 3033; Nucl. Phys. B897 (2015) 612; Phys. Rev. D92 (2015) 11405
JB, G. Falcioni, A. De Freitas DESY 15-171

Publications: Mathematics

JB, Comput. Phys. Commun. 159 (2004) 19
JB, Comput. Phys. Commun. 180 (2009) 2143; 0901.0837
JB, D. Broadhurst, J. Vermaseren, Comput. Phys. Commun. 181 (2010) 582
JB, M. Kauers, C. Schneider, Comput. Phys. Commun. 180 (2009) 2143
JB, S.Klein, C. Schneider, F. Stan. J. Symbolic Comput. 47 (2012) 1267
J. Ablinger, JB, C. Schneider, J. Math. Phys. 52 (2011) 102301, J. Math. Phys. 54 (2013) 082301
J. Ablinger, JB, 1304.7071 [Contr. to a Book: Springer, Wien]
J. Ablinger, JB, C. Raab, C. Schneider, J. Math. Phys. 55 (2014) 112301
J. Ablinger, A. Behring, JB, A. De Freitas, A. von Manteuffel, C. Schneider, Comp. Phys. Commun. 202 (2016) 33

Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x,Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)}_{perturbative} \otimes \underbrace{f_j(x,\mu^2)}_{nonpert.}$$

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs).

$\otimes$ denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz\, \delta(x - yz) f(y) g(z)\ .$$

The subsequent calculations are performed in Mellin space, where $\otimes$ reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) = \int_0^1 dx\, x^{N-1} f(x)\ .$$

Wilson coefficients:

$$\mathbb{C}_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) = C_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right) + H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) .$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) = \sum_i C_{i,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right) A_{ij}\left(\frac{m^2}{\mu^2},N\right)$$

[Buza, Matiounine, Smith, van Neerven 1996 Nucl.Phys.B]

factorizes into the light flavor Wilson coefficients C and the massive operator matrix elements (OMEs) of local operators O_i between partonic states j

$$A_{ij}\left(\frac{m^2}{\mu^2},N\right) = \langle j \mid O_i \mid j \rangle .$$

→ additional Feynman rules with local operator insertions for partonic matrix elements.

The unpolarized light flavor Wilson coefficients are known up to NNLO

[Moch, Vermaseren, Vogt, 2005 Nucl.Phys.B].

For $F_2(x,Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

Status of OME calculations

Leading Order: [Witten 1976, Babcock, Sivers, Wolfram 1978, Shifman, Vainshtein, Zakharov 1978, Leveille, Weiler 1979, Glück, Reya 1979, Glück, Hoffmann, Reya 1982]

Next-to-Leading Order:

[Laenen, van Neerven, Riemersma, Smith 1993]

$Q^2 \gg m^2$: via IBP [Buza, Matiounine, Smith, Migneron, van Neerven 1996]

Compact results via ${}_pF_q$'s [Bierenbaum, Blümlein, Klein, 2007]

$O(\alpha_s^2 \varepsilon)$ (for general N) [Bierenbaum, Blümlein, Klein 2008, 2009]

Next-to-Next-to-Leading Order: $Q^2 \gg m^2$

- Moments for F_2: $N = 2...10(14)$ [Bierenbaum, Blümlein, Klein 2009] mapping large expressions to [MATAD, Steinhauser 2000]
- Contributions to transversity: $N = 1...13$ [Blümlein, Klein, Tödtli 2009]
- Two masses $m_1 \neq m_2 \rightarrow$ Moments N $= 2,4,6$ [JB, Wißbrock 2011]

At 3-loop order for general values of N:

- All OMEs: terms $O(n_f T_F^2 C_{A/F})$ to F_2 [Ablinger et al. 2011, 2012]
- First contributions to $O(T_F^2 C_{A/F})$ $A_{gg,Q}$ [Ablinger et al. 2014]

The Wilson Coefficients at large Q^2

$$
\begin{aligned}
L_{q,(2,L)}^{\mathsf{NS}}(N_F+1) &= a_s^2\Big[A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)\ \delta_2 + \hat{C}_{q,(2,L)}^{(2),\mathsf{NS}}(N_F)\Big] \\
&+ a_s^3\Big[A_{qq,Q}^{(3),\mathsf{NS}}(N_F+1)\ \delta_2 + A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1) + \hat{C}_{q,(2,L)}^{(3),\mathsf{NS}}(N_F)\Big]
\end{aligned}
$$

2010

$$
L_{q,(2,L)}^{\mathsf{PS}}(N_F+1) = a_s^3\Big[\ A_{qq,Q}^{(3),\mathsf{PS}}(N_F+1)\ \delta_2 + A_{gq,Q}^{(2)}(N_F)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F)\Big]
$$

$$
\begin{aligned}
L_{g,(2,L)}^{\mathsf{S}}(N_F+1) &= a_s^2 A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + a_s^3\Big[\ A_{gg,Q}^{(3)}(N_F+1)\ \delta_2 \\
&+A_{gg,Q}^{(1)}(N_F+1)\ \ N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1) + A_{gg,Q}^{(2)}(N_F+1)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&+ A_{Qg}^{(1)}(N_F+1)\ \ N_F\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1) + N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\Big]\ , \\
H_{q,(2,L)}^{\mathsf{PS}}(N_F+1) &= a_s^2\Big[\ A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\ \delta_2 +\ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big] + a_s^3\Big[\ A_{Qq}^{(3),\mathsf{PS}}(N_F+1)\ \delta_2 \\
&+ \tilde{C}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F+1) + A_{gq,Q}^{(2)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&+A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)\Big]\ , \\
H_{g,(2,L)}^{\mathsf{S}}(N_F+1) &= a_s\Big[\ A_{Qg}^{(1)}(N_F+1)\ \delta_2 +\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\Big] + a_s^2\Big[\ A_{Qg}^{(2)}(N_F+1)\ \delta_2 \\
&+ A_{Qg}^{(1)}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1) +\ A_{gg,Q}^{(1)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&+ \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\Big] +\ a_s^3\Big[\ A_{Qg}^{(3)}(N_F+1)\ \delta_2 +\ A_{Qg}^{(2)}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1) \\
&+ A_{gg,Q}^{(2)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) +\ A_{Qg}^{(1)}(N_F+1)\Big\{C_{q,(2,L)}^{(2),\mathsf{NS}}(N_F+1) \\
&+ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big\} +\ A_{gg,Q}^{(1)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+1) +\ \tilde{C}_{g,(2,L)}^{(3)}(N_F+1)\Big]
\end{aligned}
$$

[Ablinger et al., 2010]

The Wilson Coefficients at large Q^2

$$
\begin{aligned}
L_{q,(2,L)}^{\mathsf{NS}}(N_F+1) &= a_s^2\left[A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)\ \delta_2 + \hat{C}_{q,(2,L)}^{(2),\mathsf{NS}}(N_F)\right] \\
&+ a_s^3\left[A_{qq,Q}^{(3),\mathsf{NS}}(N_F+1)\ \delta_2 + A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1) + \hat{C}_{q,(2,L)}^{(3),\mathsf{NS}}(N_F)\right]
\end{aligned}
$$

2010

$$
L_{q,(2,L)}^{\mathsf{PS}}(N_F+1) = a_s^3\left[\ A_{qq,Q}^{(3),\mathsf{PS}}(N_F+1)\ \delta_2 + A_{gq,Q}^{(2)}(N_F)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F)\right]
$$

2010

$$
\begin{aligned}
L_{g,(2,L)}^{\mathsf{S}}(N_F+1) &= a_s^2 A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + a_s^3\left[\ A_{qg,Q}^{(3)}(N_F+1)\ \delta_2\right. \\
&+A_{gg,Q}^{(1)}(N_F+1)\ \ N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1) + A_{gg,Q}^{(2)}(N_F+1)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&\left.+\ A_{Qg}^{(1)}(N_F+1)\ \ N_F\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1) + N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\right]\ , \\
H_{q,(2,L)}^{\mathsf{PS}}(N_F+1) &= a_s^2\left[\ A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\ \delta_2 +\ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\right] + a_s^3\left[\ A_{Qq}^{(3),\mathsf{PS}}(N_F+1)\ \delta_2\right. \\
&+\ \tilde{C}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F+1) + A_{gq,Q}^{(2)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&\left.+A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)\right]\ , \\
H_{g,(2,L)}^{\mathsf{S}}(N_F+1) &= a_s\left[\ A_{Qg}^{(1)}(N_F+1)\ \delta_2 +\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\right] + a_s^2\left[\ A_{Qg}^{(2)}(N_F+1)\ \delta_2\right. \\
&+\ A_{Qg}^{(1)}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1) +\ A_{gg,Q}^{(1)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&\left.+\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\right] +\ a_s^3\left[\ A_{Qg}^{(3)}(N_F+1)\ \delta_2 +\ A_{Qg}^{(2)}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)\right. \\
&+\ A_{gg,Q}^{(2)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) +\ A_{Qg}^{(1)}(N_F+1)\left\{C_{q,(2,L)}^{(2),\mathsf{NS}}(N_F+1)\right. \\
&\left.+\ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\right\} +\ A_{gg,Q}^{(1)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+1) +\ \left.\tilde{C}_{g,(2,L)}^{(3)}(N_F+1)\right]
\end{aligned}
$$

[Ablinger et al., 2010]

The Wilson Coefficients at large Q^2

$$
\begin{aligned}
\text{2014}\quad L_{q,(2,L)}^{\mathsf{NS}}(N_F+1) &= a_s^2\Big[A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)\;\delta_2+\hat{C}_{q,(2,L)}^{(2),\mathsf{NS}}(N_F)\Big] \\
&+ a_s^3\Big[A_{qq,Q}^{(3),\mathsf{NS}}(N_F+1)\;\delta_2+A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)+\hat{C}_{q,(2,L)}^{(3),\mathsf{NS}}(N_F)\Big] \\
\text{2010}\quad L_{q,(2,L)}^{\mathsf{PS}}(N_F+1) &= a_s^3\Big[\;A_{qq,Q}^{(3),\mathsf{PS}}(N_F+1)\;\delta_2+A_{gq,Q}^{(2)}(N_F)\;\;N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)+N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F)\Big] \\
\text{2010}\quad L_{g,(2,L)}^{\mathsf{S}}(N_F+1) &= a_s^2A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)+a_s^3\Big[\;A_{qg,Q}^{(3)}(N_F+1)\;\delta_2 \\
&\quad +A_{gg,Q}^{(1)}(N_F+1)\;\;N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)+A_{gg,Q}^{(2)}(N_F+1)\;\;N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&\quad +\;A_{Qg}^{(1)}(N_F+1)\;\;N_F\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)+N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\Big]\;, \\
H_{q,(2,L)}^{\mathsf{PS}}(N_F+1) &= a_s^2\Big[\;A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\;\delta_2+\;\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big]+a_s^3\Big[\;A_{Qq}^{(3),\mathsf{PS}}(N_F+1)\;\delta_2 \\
&\quad +\;\tilde{C}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F+1)+A_{gq,Q}^{(2)}(N_F+1)\;\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&\quad +A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\;C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)\Big]\;, \\
H_{g,(2,L)}^{\mathsf{S}}(N_F+1) &= a_s\Big[\;A_{Qg}^{(1)}(N_F+1)\;\delta_2+\;\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\Big]+a_s^2\Big[\;A_{Qg}^{(2)}(N_F+1)\;\delta_2 \\
&\quad +\;A_{Qg}^{(1)}(N_F+1)\;C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)+\;A_{gg,Q}^{(1)}(N_F+1)\;\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) \\
&\quad +\;\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\Big]+\;a_s^3\Big[\;A_{Qg}^{(3)}(N_F+1)\;\delta_2+\;A_{Qg}^{(2)}(N_F+1)\;C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1) \\
&\quad +\;A_{gg,Q}^{(2)}(N_F+1)\;\tilde{C}_{g,(2,L)}^{(1)}(N_F+1)+\;A_{Qg}^{(1)}(N_F+1)\Big\{C_{q,(2,L)}^{(2),\mathsf{NS}}(N_F+1) \\
&\quad +\;\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big\}+\;A_{gg,Q}^{(1)}(N_F+1)\;\tilde{C}_{g,(2,L)}^{(2)}(N_F+1)+\;\tilde{C}_{g,(2,L)}^{(3)}(N_F+1)\Big]
\end{aligned}
$$

[Ablinger et al. 2010, Ablinger et al., 2014a]

The Wilson Coefficients at large Q^2

2014 $$L_{q,(2,L)}^{\mathsf{NS}}(N_F+1) = a_s^2\Big[A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)\ \delta_2 + \hat{C}_{q,(2,L)}^{(2),\mathsf{NS}}(N_F)\Big] + a_s^3\Big[A_{qq,Q}^{(3),\mathsf{NS}}(N_F+1)\ \delta_2 + A_{qq,Q}^{(2),\mathsf{NS}}(N_F+1)C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1) + \hat{C}_{q,(2,L)}^{(3),\mathsf{NS}}(N_F)\Big]$$

2010 $$L_{q,(2,L)}^{\mathsf{PS}}(N_F+1) = a_s^3\Big[\ A_{qq,Q}^{(3),\mathsf{PS}}(N_F+1)\ \delta_2 + A_{gq,Q}^{(2)}(N_F)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + N_F\hat{\tilde{C}}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F$$

2010 $$L_{g,(2,L)}^{\mathsf{S}}(N_F+1) = a_s^2A_{gg,Q}^{(1)}(N_F+1)N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + a_s^3\Big[\ A_{qg,Q}^{(3)}(N_F+1)\ \delta_2 + A_{gg,Q}^{(1)}(N_F+1)\ \ N_F\tilde{C}_{g,(2,L)}^{(2)}(N_F+1) + A_{gg,Q}^{(2)}(N_F+1)\ \ N_F\tilde{C}_{g,(2,L)}^{(1)}(N_F+ + A_{Qg}^{(1)}(N_F+1)\ \ N_F\tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1) + N_F\hat{\tilde{C}}_{g,(2,L)}^{(3)}(N_F)\Big]\ ,$$

2014 $$H_{q,(2,L)}^{\mathsf{PS}}(N_F+1) = a_s^2\Big[\ A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\ \delta_2 + \ \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big] + a_s^3\Big[\ A_{Qq}^{(3),\mathsf{PS}}(N_F+1)\ \delta_2 + \tilde{C}_{q,(2,L)}^{(3),\mathsf{PS}}(N_F+1) + A_{gq,Q}^{(2)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + A_{Qq}^{(2),\mathsf{PS}}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1)\Big]\ ,$$

$$H_{g,(2,L)}^{\mathsf{S}}(N_F+1) = a_s\Big[\ A_{Qg}^{(1)}(N_F+1)\ \delta_2 + \ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1)\Big] + a_s^2\Big[\ A_{Qg}^{(2)}(N_F+1)\ \delta_2 + A_{Qg}^{(1)}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+1) + \ A_{gg,Q}^{(1)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \tilde{C}_{g,(2,L)}^{(2)}(N_F+1)\Big] + \ a_s^3\Big[\ A_{Qg}^{(3)}(N_F+1)\ \delta_2 + \ A_{Qg}^{(2)}(N_F+1)\ C_{q,(2,L)}^{(1),\mathsf{NS}}(N_F+ + A_{gg,Q}^{(2)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(1)}(N_F+1) + \ A_{Qg}^{(1)}(N_F+1)\Big\{C_{q,(2,L)}^{(2),\mathsf{NS}}(N_F+1) + \tilde{C}_{q,(2,L)}^{(2),\mathsf{PS}}(N_F+1)\Big\} + \ A_{gg,Q}^{(1)}(N_F+1)\ \tilde{C}_{g,(2,L)}^{(2)}(N_F+1) + \ \tilde{C}_{g,(2,L)}^{(3)}(N_F+1$$

[Ablinger et al. 2010, Ablinger et al., 2014a, Ablinger et al., 2014b]

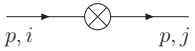
Variable Flavor Number Scheme

$$
\begin{aligned}
f_k(n_f+1,\mu^2) + f_{\bar{k}}(n_f+1,\mu^2) &= A_{qq,Q}^{\rm NS}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \left[f_k(n_f,\mu^2) + f_{\bar{k}}(n_f,\mu^2)\right] \\
&\quad + \tilde{A}_{qq,Q}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f,\mu^2) + \tilde{A}_{qg,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes G(n_f,\mu^2) \\
f_{Q+\bar{Q}}(n_f+1,\mu^2) &= \tilde{A}_{Qq}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f,\mu^2) + \tilde{A}_{Qg}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes G(n_f,\mu^2)\,. \\
G(n_f+1,\mu^2) &= A_{gq,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f,\mu^2) + A_{gg,Q}^{S}\left(n_f,\frac{\mu^2}{m^2}\right) \otimes G(n_f,\mu^2)\,. \\
\Sigma(n_f+1,\mu^2) &= \sum_{k=1}^{n_f+1}\left[f_k(n_f+1,\mu^2) + f_{\bar{k}}(n_f+1,\mu^2)\right] \\
&= \left[A_{qq,Q}^{\rm NS}\left(n_f,\frac{\mu^2}{m^2}\right) + n_f\tilde{A}_{qq,Q}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right) + \tilde{A}_{Qq}^{\rm PS}\left(n_f,\frac{\mu^2}{m^2}\right)\right] \\
&\quad \otimes\Sigma(n_f,\mu^2) \\
&\quad + \left[n_f\tilde{A}_{qg,Q}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right) + \tilde{A}_{Qg}^{\rm S}\left(n_f,\frac{\mu^2}{m^2}\right)\right] \otimes G(n_f,\mu^2)
\end{aligned}
$$

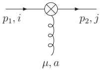
All master integrals for $A_{gg}^{(3)}$ have been completed (June 2015).

Calculation of the 3-loop operator matrix elements

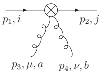
The OMEs are calculated using the QCD Feynman rules together with the following operator insertion Feynman rules:



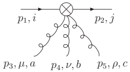
$$\delta^{ij} \not{\Delta} \gamma_\pm (\Delta \cdot p)^{N-1} , \quad N \geq 1$$



$$g t^a_{ji} \Delta^\mu \not{\Delta} \gamma_\pm \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2} , \quad N \geq 2$$



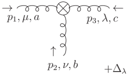
$$g^2 \Delta^\mu \Delta^\nu \not{\Delta} \gamma_\pm \sum_{j=0}^{N-3} \sum_{l=j+1}^{N-2} (\Delta p_2)^j (\Delta p_1)^{N-l-2}$$
$$\left[(t^a t^b)_{ji} (\Delta p_1 + \Delta p_4)^{l-j-1} + (t^b t^a)_{ji} (\Delta p_1 + \Delta p_3)^{l-j-1} \right] ,$$
$$N \geq 3$$



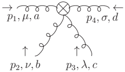
$$g^3 \Delta_\mu \Delta_\nu \Delta_\rho \not{\Delta} \gamma_\pm \sum_{j=0}^{N-4} \sum_{l=j+1}^{N-3} \sum_{m=l+1}^{N-2} (\Delta.p_2)^j (\Delta.p_1)^{N-m-2}$$
$$\Big[\; (t^a t^b t^c)_{ji} (\Delta.p_4 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_5 + \Delta.p_1)^{m-l-1}$$
$$+ (t^a t^c t^b)_{ji} (\Delta.p_4 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_4 + \Delta.p_1)^{m-l-1}$$
$$+ (t^b t^a t^c)_{ji} (\Delta.p_3 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_5 + \Delta.p_1)^{m-l-1}$$
$$+ (t^b t^c t^a)_{ji} (\Delta.p_3 + \Delta.p_5 + \Delta.p_1)^{l-j-1} (\Delta.p_3 + \Delta.p_1)^{m-l-1}$$
$$+ (t^c t^a t^b)_{ji} (\Delta.p_3 + \Delta.p_4 + \Delta.p_1)^{l-j-1} (\Delta.p_4 + \Delta.p_1)^{m-l-1}$$
$$+ (t^c t^b t^a)_{ji} (\Delta.p_3 + \Delta.p_4 + \Delta.p_1)^{l-j-1} (\Delta.p_3 + \Delta.p_1)^{m-l-1} \Big] ,$$
$$N \geq 4$$

$$\gamma_+ = 1 , \quad \gamma_- = \gamma_5 .$$

$$\frac{1+(-1)^N}{2} \delta^{ab} (\Delta \cdot p)^{N-2}$$
$$\left[g_{\mu\nu} (\Delta \cdot p)^2 - (\Delta_\mu p_\nu + \Delta_\nu p_\mu) \Delta \cdot p + p^2 \Delta_\mu \Delta_\nu \right] , \quad N \geq 2$$



$$-ig \frac{1+(-1)^N}{2} f^{abc} \Bigg($$
$$\left[(\Delta_\nu g_{\lambda\mu} - \Delta_\lambda g_{\mu\nu}) \Delta \cdot p_1 + \Delta_\mu (p_{1,\nu} \Delta_\lambda - p_{1,\lambda} \Delta_\nu) \right] (\Delta \cdot p_1)^{N-2}$$
$$+ \Delta_\lambda \left[\Delta \cdot p_1 p_{2,\mu} \Delta_\nu + \Delta \cdot p_2 p_{1,\nu} \Delta_\mu - \Delta \cdot p_1 \Delta \cdot p_2 g_{\mu\nu} - p_1 \cdot p_2 \Delta_\mu \Delta_\nu \right]$$
$$\times \sum_{j=0}^{N-3} (-\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-3-j}$$
$$+ \left\{ \begin{smallmatrix} p_1 \to p_2 \to p_3 \to p_1 \\ \mu \to \nu \to \lambda \to \mu \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} p_1 \to p_3 \to p_2 \to p_1 \\ \mu \to \lambda \to \nu \to \mu \end{smallmatrix} \right\} \Bigg) , \quad N \geq 2$$



$$g^2 \frac{1+(-1)^N}{2} \Bigg(f^{abe} f^{cde} O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4)$$
$$+ f^{ace} f^{bde} O_{\mu\lambda\nu\sigma}(p_1, p_3, p_2, p_4) + f^{ade} f^{bce} O_{\mu\sigma\nu\lambda}(p_1, p_4, p_2, p_3) \Bigg) ,$$
$$O_{\mu\nu\lambda\sigma}(p_1, p_2, p_3, p_4) = \Delta_\nu \Delta_\lambda \Bigg\{ -g_{\mu\sigma} (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-2}$$
$$+ [p_{4,\mu} \Delta_\sigma - \Delta \cdot p_4 g_{\mu\sigma}] \sum_{i=0}^{N-3} (\Delta \cdot p_3 + \Delta \cdot p_4)^i (\Delta \cdot p_4)^{N-3-i}$$
$$- [p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 g_{\mu\sigma}] \sum_{i=0}^{N-3} (-\Delta \cdot p_1)^i (\Delta \cdot p_3 + \Delta \cdot p_4)^{N-3-i}$$
$$+ [\Delta \cdot p_1 \Delta \cdot p_4 g_{\mu\sigma} + p_1 \cdot p_4 \Delta_\mu \Delta_\sigma - \Delta \cdot p_4 p_{1,\sigma} \Delta_\mu - \Delta \cdot p_1 p_{4,\mu} \Delta_\sigma]$$
$$\times \sum_{i=0}^{N-4} \sum_{j=0}^{i} (-\Delta \cdot p_1)^{N-4-i} (\Delta \cdot p_3 + \Delta \cdot p_4)^{i-j} (\Delta \cdot p_4)^j \Bigg\}$$
$$- \left\{ \begin{smallmatrix} p_1 \leftrightarrow p_2 \\ \mu \leftrightarrow \nu \end{smallmatrix} \right\} - \left\{ \begin{smallmatrix} p_3 \leftrightarrow p_4 \\ \lambda \leftrightarrow \sigma \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} p_1 \leftrightarrow p_2, \; p_3 \leftrightarrow p_4 \\ \mu \leftrightarrow \nu, \; \lambda \leftrightarrow \sigma \end{smallmatrix} \right\} , \quad N \geq 2$$

The diagrams are generated using QGRAF [Nogueira 1993 J. Comput. Phys].

	$A_{qq,Q}^{(3),\mathrm{NS}}$	$A_{gq,Q}^{(3)}$	$A_{Qq}^{(3),\mathrm{PS}}$	$A_{gg,Q}^{(3)}$	$A_{Qg}^{(3)}$
No. diagrams	110	86	125	642	1233

A FORM [Vermaseren 2000] program was written in order to perform the γ-matrix algebra in the numerator of all diagrams, which are then expressed as a linear combination of scalar integrals.

$$\begin{aligned} A_{qq,Q}^{(3),\mathrm{NS}} &\rightarrow \quad 7426 \text{ scalar integrals.} \\ A_{gq,Q}^{(3)} &\rightarrow \quad 12529 \text{ scalar integrals.} \\ A_{Qq}^{(3),\mathrm{PS}} &\rightarrow \quad 5470 \text{ scalar integrals.} \end{aligned}$$

$\Longrightarrow$ Need to use integration by parts identities.
$\Longrightarrow$ The reduction for all OMEs has been completed.

$\Longrightarrow$ Use special computers: 12 units with overall 3.2 TB RAM, 97 TB fast disc, hundreds of `mathematica` lic. ; IBP: several TB of final relations.

Integration by parts

We use Reduze [A. von Manteuffel, C. Studerus, 2012] to express all scalar integrals required in the calculation in terms of a small(er) set of master integrals.

Reduze is a C++ program based on Laporta's algorithm.

$$(\Delta \cdot k)^N \to \sum_{N=0}^{\infty} x^N (\Delta \cdot k)^N = \frac{1}{1 - x\Delta \cdot k}$$

$\Longrightarrow$ additional propagator.

Number of master integrals:

$$\begin{array}{rcl} A_{qq,Q}^{(3),\mathrm{NS}} & \to & 35 \text{ master integrals } \surd. \\ A_{gq,Q}^{(3)} & \to & 41 \text{ master integrals } \surd. \\ A_{Qq}^{(3),\mathrm{PS}} & \to & 66 \text{ master integrals } \surd. \\ A_{gg,Q}^{(3)} & \to & 205 \text{ master integrals } \surd. \\ A_{Qg}^{(3)} & \to & 340 \text{ master integrals. (224 done by June 2015.)} \end{array}$$

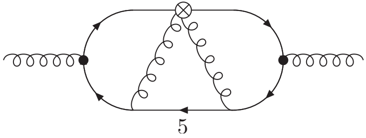
24 integral families are required and implemented in Reduze.

Calculation of the master integrals

For the calculation of the master integrals we use a wide variety of tools:

- Hypergeometric functions.
- Summation methods based on difference fields, implemented in the `Mathematica` program `Sigma` [C. Schneider, 2005–].
  - Reduction of the sums to a small number of key sums.
  - Expansion the summands in ε.
  - Simplification by symbolic summation algorithms based on $\Pi\Sigma$-fields [Karr 1981 J. ACM, Schneider 2005–].
  - Harmonic sums, polylogarithms and their various generalizations are algebraically reduced using the package HarmonicSums [Ablinger 2010, 2013, Ablinger, Blümlein, Schneider 2011,2013].
- Mellin-Barnes representations.
- In the case of convergent massive 3-loop Feynman integrals, they can be performed in terms of Hyperlogarithms [Generalization of a method by F. Brown, 2008, to non-vanishing masses and local operators].
- Systems of Differential Equations.
- Almkvist-Zeilberger Theorem as Integration Method.

V-Topology



- Emergence of a new function class : nested generalized cyclotomic sums, weighted with binomials and inverse binomials of the type $\binom{2i}{i}$.
- At the side of the iterated integrals many root-valued letters appear (around 30).
- The scalar diagram exhibits terms growing like $8^N, 4^N, 2^N, N \to \infty$. The growth 2^N survives in the scalar case. The physical diagram is free of this divergence.
- Asymptotic representations can be constructed analytically to arbitrary precision.
- Various special new numbers appear, the simplest of which is π, through which ζ_2 is no longer and elementary constant here.

Emergence of new nested sums :

$$\sum_{i=1}^{N}\binom{2i}{i}(-2)^i\sum_{j=1}^{i}\frac{1}{j\binom{2j}{j}}S_{1,2}\left(\tfrac{1}{2},-1;j\right)$$
$$=\int_0^1 dx\frac{x^N-1}{x-1}\sqrt{\frac{x}{8+x}}\left[\mathrm{H}^*_{w_{17},-1,0}(x)-2\mathrm{H}^*_{w_{18},-1,0}(x)\right]$$
$$+\frac{\zeta_2}{2}\int_0^1 dx\frac{(-x)^N-1}{x+1}\sqrt{\frac{x}{8+x}}\left[\mathrm{H}^*_{12}(x)-2\mathrm{H}^*_{13}(x)\right]$$
$$+c_3\int_0^1 dx\frac{(-8x)^N-1}{x+\frac{1}{8}}\sqrt{\frac{x}{1-x}}\,,$$

$$w_{12}=\frac{1}{\sqrt{x(8-x)}},\qquad w_{13}=\frac{1}{(2-x)\sqrt{x(8-x)}},$$
$$w_{17}=\frac{1}{\sqrt{x(8+x)}},\qquad w_{18}=\frac{1}{(2+x)\sqrt{x(8+x)}}\,.$$

$\sim$ 100 associated independent nested sums. The associated iterated integrals request root-valued alphabets with about 30 new letters.

[J. Ablinger, J. Bümlein, J. Raab, C. Schneider, F. Wißbrock 2014.]

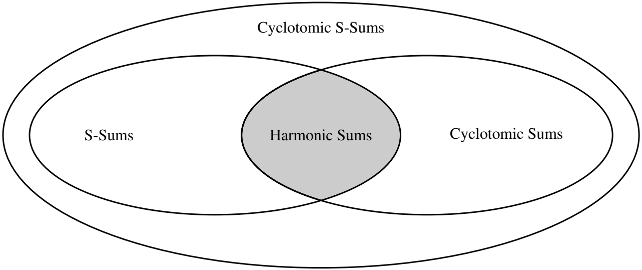
[J. Ablinger, J. Bümlein, J. Raab, C. Schneider 2014.]

Spill-Off:
New Mathematical Function Classes and Algebras

- 1998: Harmonic Sums [Vermaseren; JB]
- 1999: Harmonic Polylogarithms [Remiddi, Vermaseren]
- 2001: Generalized Harmonic Sums [Moch, Uwer, Weinzierl]
- 2004: Infinite harmonic (inverse) binomial sums [Davydychev, Kalmykov; Weinzierl]
- 2011: (generalized) Cyclotomic Harmonic Sums, polylogarithms and numbers [Ablinger, JB, Schneider]
- 2013: Systematic Theory of Generalized Harmonic Sums, polylogarithms and numbers [Ablinger, JB, Schneider]
- 2014: Finite nested Generalized Cyclotomic Harmonic Sums with (inverse) Binomial Weights [Ablinger, JB, Raab, Schneider]
- 2016: Elliptic integrals with (involved) rational arguments appear in part of the functions of our project already as base cases. They stem from Heun equations. [last week.]

Particle Physics Generates NEW Mathematics.

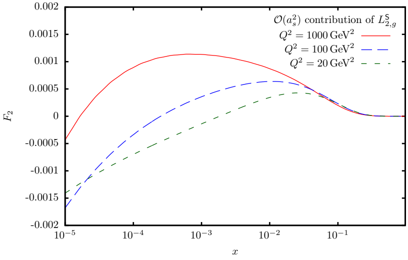
1



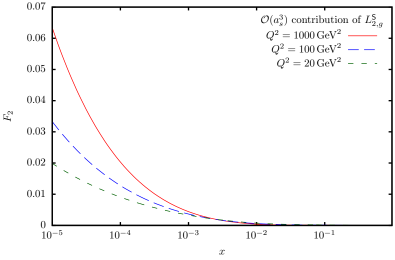
Nested (inverse) binomial sums ....

More and more onion skins to be added during theses calculations.

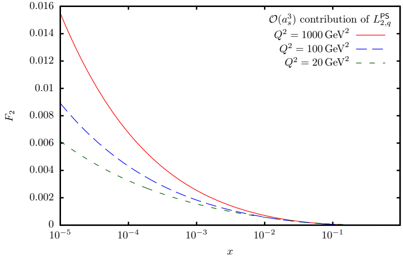
Numerical Results : $L^{\mathrm{S}}_{g,2}$ and $L^{\mathrm{PS}}_{q,2}$



$O(a_s^2)$ $L^{\mathrm{S}}_{2,g}$

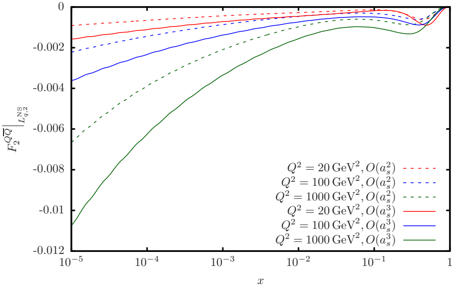


$O(a_s^3)$ $L^{\mathrm{S}}_{2,g}$

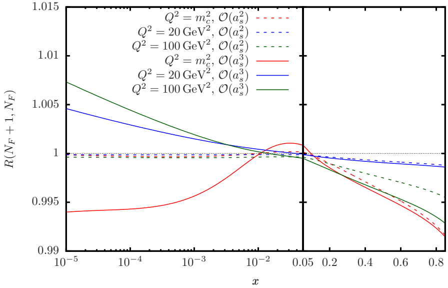


$L^{\mathrm{PS}}_{q,2}$

$L_{q,2}^{\rm NS}$

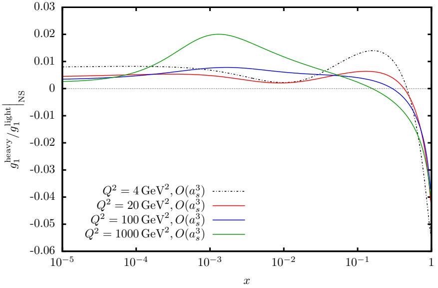


Contribution to $F_2(x, Q^2)$

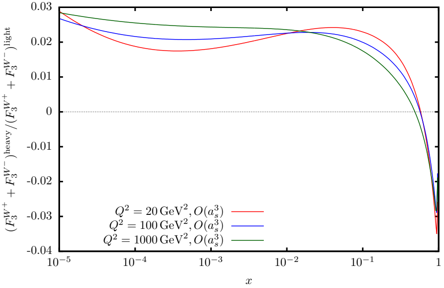


VFNS matching

NS corrections to $g_{1(2)}(x, Q^2)$ and $xF_3^{W^+ + W^-}$



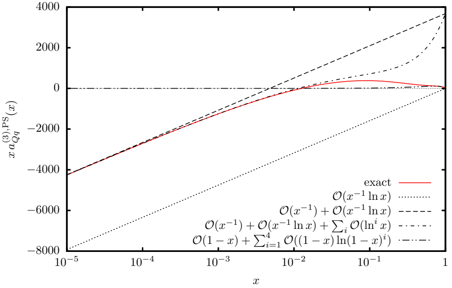
$g_1(x, Q^2)$



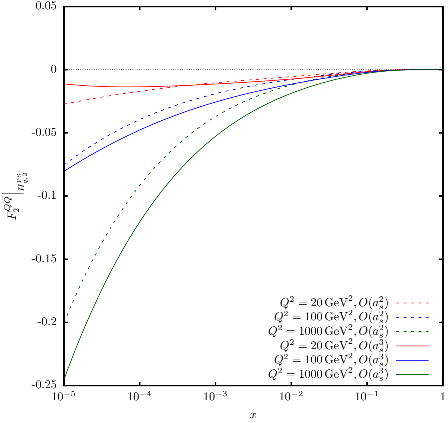
$xF_3^{W^+ + W^-}(x, Q^2)$

The corrections to $g_2(x, Q^2)$ are obtained using the Wandzura-Wilczek relation.

$H_{q,2}^{\mathrm{PS}}$



$a_{Qq}^{(3),\mathrm{PS}}$



Contribution to $F_2(x, Q^2)$

The leading small x approximation corresponding to CCH, 1991, departs from the physical result everywhere except for $x = 1$.

3-Loop OME: $A_{gg,Q}$

$$
\begin{aligned}
a_{gg,Q}^{(3)} = \frac{1+(-1)^N}{2} \Biggl\{ & C_F^2 T_F \Biggl[\frac{16(N^2+N+2)}{N^2(N+1)^2} \sum_{i=1}^{N} \frac{\binom{2i}{i}\left(\sum_{j=1}^{i} \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} - 7\zeta_3\right)}{4^i (i+1)^2} - \frac{4 P_{69} S_1^2}{3(N-1)N^4(N+1)^4(N+2)} \\
& + \tilde{\gamma}_{gq}^{(0)} \left(\frac{128\left(S_{-4} - S_{-3}S_1 + S_{-3,1} + 2S_{-2,2}\right)}{3N(N+1)(N+2)} + \frac{4(5N^2+5N-22)S_1^2 S_2}{3N(N+1)(N+2)} + \cdots \right) + \cdots \Biggr] \\
& + C_A C_F T_F \Biggl[\frac{16 P_{42}}{3(N-1)N^2(N+1)^2(N+2)} \sum_{i=1}^{N} \frac{\binom{2i}{i}\left(\sum_{j=1}^{i} \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} - 7\zeta_3\right)}{4^i (i+1)^2} + \frac{32 P_2 S_{-2,2}}{(N-1)N^2(N+1)^2(N+2)} \\
& - \frac{64 P_{14} S_{-2,1,1}}{3(N-1)N^2(N+1)^2(N+2)} - \frac{16 P_{23} S_{-4}}{3(N-1)N^2(N+1)^2(N+2)} + \frac{4 P_{63} S_4}{3(N-2)(N-1)N^2(N+1)^2(N+2)} + \cdots \Biggr] \\
& + C_A^2 T_F \Biggl[- \frac{4 P_{46}}{3(N-1)N^2(N+1)^2(N+2)} \sum_{i=1}^{N} \frac{\binom{2i}{i}\left(\sum_{j=1}^{i} \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} - 7\zeta_3\right)}{4^i (i+1)^2} + \frac{256 P_5 S_{-2,2}}{9(N-1)N^2(N+1)^2(N+2)} \\
& + \frac{32 P_{30} S_{-2,1,1} + 16 P_{35} S_{-3,1} + 16 P_{44} S_{-4}}{9(N-1)N^2(N+1)^2(N+2)} + \frac{16 P_{52} S_{-2}^2}{27(N-1)N^2(N+1)^2(N+2)} + \frac{8 P_{36} S_2^2}{9(N-1)N^2(N+1)^2} + \cdots \Biggr] \\
& + C_F T_F^2 \Biggl[- \frac{16 P_{48} \binom{2N}{N} 4^{-N} \left(\sum_{i=1}^{N} \frac{4^i S_1(i-1)}{\binom{2i}{i} i^2} - 7\zeta_3 \right)}{3(N-1)N(N+1)^2(N+2)(2N-3)(2N-1)} - \frac{32 P_{86} S_1}{81(N-1)N^4(N+1)^4(N+2)(2N-3)(2N-1)} \\
& + \frac{16 P_{45} S_1^2}{27(N-1)N^3(N+1)^3(N+2)} - \frac{16 P_{45} S_2}{9(N-1)N^3(N+1)^3(N+2)} + \cdots \Biggr] + \cdots \Biggr\} \qquad (1)
\end{aligned}
$$

Also, with this calculation we were able to rederive the three loop anomalous dimension $\gamma_{gg}^{(3)}$ for the terms $\propto T_F$, and obtained agreement with the literature.

Moments for graphs with two massive lines ($m_1 \neq m_2$)

$$
\begin{aligned}
a_{Qg}^{(3)}(N=6) \;=\; & \frac{1}{2}\Bigg\{T_F^2 C_A\Bigg\{\frac{69882273800453}{367569090000} - \frac{395296}{19845}\zeta_3 + \frac{1316809}{39690}\zeta_2 + \frac{832369820129}{14586075000}x + \frac{1511074426112}{624023544375}x^2 - \frac{84840004938801319}{690973782403905000}x^3 \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right)\left[\frac{11771644229}{194481000} + \frac{78496}{2205}\zeta_2 - \frac{1406143531}{69457500}x - \frac{105157957}{180093375}x^2 + \frac{2287164970759}{7669816654500}x^3\right] \\
& + \ln^2\left(\frac{m_2^2}{\mu^2}\right)\left[\frac{2668087}{79380} + \frac{112669}{661500}x - \frac{49373}{51975}x^2 - \frac{31340489}{34054020}x^3\right] + \ln^3\left(\frac{m_2^2}{\mu^2}\right)\frac{324148}{19845} + \ln^2\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\frac{156992}{6615} \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{128234}{3969} - \frac{112669}{330750}x + \frac{98746}{51975}x^2 + \frac{31340489}{17027010}x^3\right] + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln^2\left(\frac{m_1^2}{\mu^2}\right)\frac{68332}{6615} \\
& + \ln\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{83755534727}{583443000} + \frac{78496}{2205}\zeta_2 + \frac{1406143531}{69457500}x + \frac{105157957}{180093375}x^2 - \frac{2287164970759}{7669816654500}x^3\right] \\
& + \ln^2\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{2668087}{79380} + \frac{112669}{661500}x - \frac{49373}{51975}x^2 - \frac{31340489}{34054020}x^3\right] + \ln^3\left(\frac{m_1^2}{\mu^2}\right)\frac{412808}{19845}\Bigg\} \\
& + T_F^2 C_F\Bigg\{-\frac{3161811182177}{71471767500} + \frac{447392}{19845}\zeta_3 + \frac{9568018}{4862025}\zeta_2 - \frac{64855635472}{2552563125}x + \frac{1048702178522}{97070329125}x^2 + \frac{1980566069882672}{2467763508585375}x^3 \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right)\left[\frac{1786067629}{204205050} - \frac{111848}{15435}\zeta_2 - \frac{128543024}{24310125}x - \frac{22957168}{3361743}x^2 - \frac{2511536080}{2191376187}x^3\right] \\
& + \ln^2\left(\frac{m_2^2}{\mu^2}\right)\left[\frac{3232799}{4862025} + \frac{752432}{231525}x + \frac{177944}{40425}x^2 + \frac{127858928}{42567525}x^3\right] - \ln^3\left(\frac{m_2^2}{\mu^2}\right)\frac{111848}{19845} - \ln^2\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\frac{223696}{46305} \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{22238456}{4862025} - \frac{1504864}{231525}x - \frac{355888}{40425}x^2 - \frac{255717856}{42567525}x^3\right] + \ln\left(\frac{m_2^2}{\mu^2}\right)\ln^2\left(\frac{m_1^2}{\mu^2}\right)\frac{223696}{46305} \\
& + \ln\left(\frac{m_1^2}{\mu^2}\right)\left[-\frac{24797875607}{1021025250} - \frac{111848}{15435}\zeta_2 + \frac{128543024}{24310125}x + \frac{22957168}{3361743}x^2 + \frac{2511536080}{2191376187}x^3\right] \\
& + \ln^2\left(\frac{m_1^2}{\mu^2}\right)\left[\frac{3232799}{4862025} + \frac{752432}{231525}x + \frac{177944}{40425}x^2 + \frac{127858928}{42567525}x^3\right] - \ln^3\left(\frac{m_1^2}{\mu^2}\right)\frac{1230328}{138915}\Bigg\}\Bigg\} + O\left(x^4 \ln^3(x)\right)
\end{aligned}
$$

$\rightarrow$ `q2e/exp` [Harlander, Seidensticker, Steinhauser 1999] $x = m_1^2/m_2^2$

Analytic general N results are available for $A_{qq,Q}^{\mathrm{NS}}$, A_{Qq}^{PS} and the scalar integrals of $A_{gg,Q}$.

Conclusions

- 2009: 10-14 Mellin Moments for all massive 3-loop OMEs, WC. 2010: Wilson Coefficients $L_q^{(3),\mathrm{PS}}(N)$, $L_g^{(3),\mathrm{S}}(N)$.
- 2013: Ladder, V-Graph and Benz-topologies for graphs, with no singularities in ε can be systematically calculated for general N.
- Here new functions occur (including a larger number of root-letters in iterated integrals).
- 2014 $L_q^{\mathrm{NS},(3)}$, $A_{gq,Q}^{\mathrm{S},(3)}$, $A_{qq,Q}^{\mathrm{NS,TR}(3)}$, $H_{2,q}^{\mathrm{PS}(3)}$ and $A_{Qq}^{\mathrm{PS}(3)}$ were completed.
- A method for the calculation of graphs with two massive lines of equal masses and operator insertions has been developed and applied $A_{gg,Q}^{(3)}$.
- The method can be generalized to the case of unequal masses. Here the moments for $N = 2, 4, 6$ for all graphs with two quark lines of unequal masses are now known [$\longrightarrow$ extended renormalization]; for some OMEs the complete 2-mass structure has been computed.
- The $O(\alpha_s^2)$ charged current Wilson coefficients have been completed.

Conclusions

- The corresponding 3-loop anomalous dimensions were computed, those for transversity for the first time ab initio; those for the PS-case independently for the first time.
- In all NS-cases [NC and CC] we also computed all power corrections at $O(a_s^2)$ and the associated sum rules in the inclusive case improving an earlier result by JB & W. van Neerven. [→ G. Falcioni]
- All master integrals based on iterative integrals over whatsoever alphabet for $A_{gg,Q}^{(3)}$ and $A_{Qg}^{(3)}$ have been computed and $A_{gg,Q}^{(3)}$ is known for any even integer moment $N \geq 2$. Here all the topologies, including the ladder- and V-topologies have been solved.
- We have all the principal means to reconstruct $A_{Qg}^{(3)}$ systematically at very high accuracy. The full analytic solution will request more mathematical efforts.
- Different new computer-algebra and mathematical technologies were developed. These efforts will continue. The technologies are certainly useful for various present and upcoming calculations for the LHC and ILC.