From Moments to Functions in Higher Order QCD

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- Introduction
- Single Scale Feynman Integrals as Recurrent Quantities
- Establishing and Solving Recurrences
- Application to 3-Loop Anomalous Dimensions and Wilson coefficients
- Conclusions

1. Introduction

- Higher order calculations in Quantum Field Theories easily become tedious due to the larger number of terms and the sophistication of the Feynman parameter integrals.
- This even applies to Zero Scale and Single Scale Quantities.
- Even more this is the case for higher scale problems.
- While in the latter case the mathematical structure of the solution for the Feynman Integrals is widely unknown, it is explored to a certain extent for Zero Scale and Single Scale quantities.
- Zero Scale quantities are the expansion coefficients of the running couplings and masses, fixed moments of splitting functions etc.
- They can be expressed by rational numbers and certain special numbers as multiple ζ-values and related quantities.

Introduction

- Single Scale quantities depend on a scale $z \in [0, 1]$, with z a ratio of Lorentz invariants. One may perform a Mellin Transform over z

$$\int_0^1 dz z^{N-1} f(z) = M[f](N)$$

- Here one assumes $N \in \mathbf{N}, N > 0$. Due to this the problem on hand becomes discrete.
- One may seek a description in terms of difference equations.
- Zero Scale problems are obtained from Single Scale problems treating N as a fixed integer or considering the limit $N \to \infty$.

Some Remarks about MZV's

- General question on the bases of MZV's: length in the non-alternating and alternating cases.
- Do Zero Scale Feynman integrals always lead to MZV's ?
- No! e.g. Y. Andre, 2008.
- At lower orders in perturbation theory one has just MZV's even in single-mass problems.
- J.B., Broadhurst, Vermaseren, DESY 09–003: explicit calculation of bases for alternating MZV's to w=12 and non-alternating MZV's to w=22. [World Record.]; Verification to w=26.
- Broadhurst 1996 conjecture is proven. shuffles, stuffles, doubling, gen. doubling relations However, we did not find further reductions - which still may exist.

Introduction

- Can one reconstruct the general formula for Single Scale quantities out of a finite number of fixed moments ?
- This is possible for recurrent quantities.
- At least up to 3-loop order, presumably to higher orders, single scale quantities belong to this class.
- Goal : design a general formalism to solve the problem.

2. Single Scale Feynman Integrals as Recurrent Quantities

- Can one reconstruct the general formula for Single Scale quantities out of a finite number of fixed moments ?
- Polynomials and Nested Harmonic Sums obey recurrence relations, so do their polynomials.
- Example: Harmonic Sums or linear combinations thereof:

$$F(N+1) - F(N) = \frac{\text{sign}(a)^{N+1}}{(N+1)^{|a|}}$$

is solved by $S_a(N)$; and similarly for deeper nested sums

$$S_{a,\vec{b}}(N) = \sum_{k=1}^{N} \frac{(\text{sign}(a))^k}{k^{|a|}} S_{\vec{b}}(k)$$

.

Single Scale Feynman Integrals as Recurrent Quantities

- Feynman integrals have often a form like

$$\int_0^1 dz \frac{z^{N-1} - 1}{1 - z} H_{\vec{a}}(z), \qquad \int_0^1 dz \frac{(-z)^{N-1} - 1}{1 + z} H_{\vec{a}}(z)$$

- This structure leads to recurrences.
- It is very likely that single scale Feynman diagrams do always obey difference equations.

3. Establishing and Solving Recurrences

- One seeks the relation

$$\sum_{k=0}^{l} \left[\sum_{i=0}^{d} c_{i,k} N^i \right] F(N+k) = 0 \ .$$

- The corresponding linear system is dense.
- Rational number arithmetics is not applicable for the large systems to be determined; $C^{(3)}_{2,q,C_F^3}$ would require 11 Tb of memory.
- Use arithmetic in finite fields together with Chinese remaindering $\Longrightarrow$ few Gb of memory
- The linear system approximately minimizes for $l \approx d$.
- Join different recurrences found to reduce l to a minimal value.

Establishing and Solving Recurrences

- For the solution of the recurrence low degrees are clearly preferred.
- The linear difference equation of order l with polynomial coefficients is equivalent to a linear system in l variables.
- It is solved in $\Pi - \Sigma$ fields.
- Apply advanced symbolic summation methods: telescoping, creative telescoping and its refinement. Code: `sigma`.
- The solutions are found as linear combinations of rational terms in N combined with functions, which cannot be further reduced in the $\Pi - \Sigma$ fields. In the present application they turn out all to be harmonic sums $S_{\vec{b}}(N)$.
- Other or higher order applications may consist of other sums too, which are uniquely found by the algorithm.

4. Application to 3-Loop Anomalous Dimensions and Wilson coefficients

- We apply the method for the unfolding of the unpolarized anomalous dimensions and Wilson coefficients up to 3-loop order.
- $\Longrightarrow$ analyze for individual color factors; 141 contributions from $1-3$ loops
- Input: Moch, Vermaseren, Vogt, 2004/05. The expressions are given in terms of harmonic sums.
- Calculate the moments (rational numbers) recursively through recursions for the harmonic sums; MAPLE code.
- Establish the corresponding difference equation by a recurrency finder; build a difference equation of minimal order possible; test the recurrency.
- Solve the difference equation order by order with the summation package `sigma` C. Schneider.; most complicated cases: 4 weeks @ $\leq$ 10Gb, 2 GHz Proc.

Input

```
C2qq3CF^3
N=3:
#11 digits / #10 digits

-98268084191 / 1166400000

N=500:
#1262 digits / #1256 digits

16418407704241967809530206191763765062843035444812620830571976007465070084937939944
42241103234415916303114822220582876889422095708591511216773075853139951009783631799
25189528176220340371861328469746270216726780129136750995112038078119385930439108033
50443459202186960525883320363553250899983613542268823673221490376310537617643487722
54038108742649687295200756192272854718024194037272078224737659999002363837403152999
20505336016334843482494547575553446642108141111400654753911367986891674100650767499
35787094786833905739774100135208944944639092913274258157665663863972761583173877488
59454713926460897008751574450750731923285428909654620048057119987481444143793860933
99373617980290444257899537261336751997905237704272985005100634640619858400662960711
33725430156489191559640696069945973638863011850678272919370653007547869470636728488
93820819268710786003286281319367660574759704508965566676221633658958087734281197211
53527921310890635770450696939622130611988940570336060686956071232719697269810600566
01158460943602399862339178722607222773226904501323768362535491521301166456705650455
96669459201645860239580602717466067988988613607723330880307417756055465187887933277
22643682970712174056544743758442382508892385389745484212981704259095217425594947288
72017877003947396562261659860366839154407853462338171648227013134266795320251847
/
```

3057444614247225372882570514367358697278130741348282122206492932820352440850471902
7491046962105336645563654873675690796713906565688820365601907263710863954826386081
3227580037879361869941003802807590860358894142891046776447162895908787986423254678
5776778283337231702130612499429819559798501074020676282769289102955679421885795867
1982932998601320344971927374905889934059987271939760212836368619501189238215442366
3805773701929509268157747992859384837403751183019423692868569168206789710047557452
5131217382272060267681480496298975522467614707848639773185909858278799786637303834
1017166676276847525704755493166263297079720470719813623901545811953853986456533543
9994182050551827959988760121168490745476969259468454613431624179198860751513076481
0304734205926703138519418575731315944374873897873646706993620825697218523316375559
4068222004765962715924208526106008109740402380126260947524640509361283802755722132
4856690051525724685919792641506082307567956962328560073471086799287131287564668441
6256698083504233897436484702002471314330803421467773925541151273924985946178771189
2312437162213438137703896064734987157020801413153555435311326719739117599044341913
5922693587373856609594245948237469293148702516714038297077639382332251255360181047
496586232475091126597629976797375278827111116774593003520000000000000000000

N=5114:

#13388 digits / #13381 digits

Table 1: Run parameters for the unfolding of the non-singlet anomalous dimensions

	number of terms needed	order of recurrence	degree of recurrence	total time [sec]	length of recurrence [kbyte]	number of harm. sums a [b]	solution time [sec]
$P_{NS,0}$	14	2	3	0.05	0.087	1 [1]	0.55
$P^-_{NS,1,C_F^2}$	142	5	31	3.32	4.666	6 [10]	7.45
$P^-_{NS,1,C_AC_F}$	109	4	24	1.91	2.834	6 [7]	6.28
$P^-_{NS,1,C_FN_F}$	24	2	7	0.13	0.271	2 [2]	0.92
$P^+_{NS,1,C_F^2}$	142	5	31	3.35	4.707	6 [10]	7.45
$P^+_{NS,1,C_AC_F}$	109	4	23	1.88	2.703	6 [7]	6.23
$P^+_{NS,1,C_FN_F}$	24	2	7	0.09	0.271	2 [2]	0.89
$P^-_{NS,2,C_F^3}$	1079	16	192	3152.19	529.802	25 [68]	1194.41
$P^-_{NS,2,C_F^3\zeta_3}$	48	3	11	0.49	0.643	1 [1]	1.56
$P^-_{NS,2,C_AC_F^2}$	974	15	181	1736.08	450.919	25 [62]	1194.41
$P^-_{NS,2,C_AC_F^2\zeta_3}$	48	3	11	0.53	0.643	1 [1]	1.53
$P^-_{NS,2,C_A^2C_F}$	749	12	147	1004.12	242.892	25 [62]	1100.88
$P^-_{NS,2,C_A^2C_F\zeta_3}$	48	3	11	0.56	0.643	1 [1]	1.56
$P^-_{NS,2,C_FN_F^2}$	39	2	11	0.31	0.369	3 [3]	1.20
$P^-_{NS,2,C_F^2N_F}$	377	8	68	76.34	33.946	12 [24]	72.22
$P^-_{NS,2,C_F^2N_F\zeta_3}$	14	2	3	0.12	0.101	1 [1]	0.53
$P^-_{NS,2,C_AC_FN_F}$	356	7	62	65.25	23.830	12 [20]	52.67
$P^-_{NS,2,C_AC_FN_F\zeta_3}$	14	2	3	0.12	0.101	1 [1]	0.55
$P^+_{NS,2,C_F^3}$	1079	16	192	4713.27	527.094	25[68]	1165.22
$P^+_{NS,2,C_F^3\zeta_3}$	48	3	11	0.55	0.643	1[1]	1.562
$P^+_{NS,2,C_AC_F^2}$	974	15	178	1715.03	442.031	25[62]	889.047
$P^+_{NS,2,C_AC_F^2\zeta_3}$	48	3	11	0.61	0.643	1[1]	1.531
$P^+_{NS,2,C_A^2C_F}$	749	12	146	991.22	240.325	25[50]	516.812
$P^+_{NS,2,C_A^2C_F\zeta_3}$	48	3	11	0.61	0.643	1[1]	1.593
$P^+_{NS,2,C_F^2N_F}$	377	8	69	111.38	33.872	12[24]	71.235
$P^+_{NS,2,C_F^2N_F\zeta_3}$	14	2	3	0.15	0.101	1[1]	0.531
$P^+_{NS,2,C_AC_FN_F}$	307	7	61	48.62	23.808	12[24]	71.235
$P^+_{NS,2,C_AC_FN_F\zeta_3}$	14	2	3	0.15	0.101	1[1]	0.547
$P^+_{NS,2,C_FN_F^2}$	39	2	11	0.40	0.369	3[3]	1.172
$P^-_{NS,2,N_Fd_{abc}}$	39	2	11	0.55	0.369	3 [3]	1.19

Table 2: Run parameters for the unfolding of the unpolarized quarkonic Wilson Coefficients for the structure function $F_2(x, Q^2)$.

	number of terms needed	order of recurrence	degree of recurrence	total time [sec]	length of recurrence [kbyte]	number of harm. sums a [b]	solution time [sec]
$C_{2,q,C_F}^{(1)}$	35	3	7	0.26	0.429	2[3]	1.13
$C_{2,q,C_F^2}^{(2)}$	689	11	137	1134.10	177.806	13[39]	258.24
$C_{2,q,C_AC_F}^{(2)}$	545	10	121	413.33	127.893	12[35]	178.73
$C_{2,q,C_F^2\zeta_3}^{(2)}$	15	2	3	0.27	0.100	1[1]	0.54
$C_{2,q,C_AC_F\zeta_3}$	15	2	3	0.27	0.112	1[1]	0.55
C_{2,q,N_FC_F}	71	4	16	2.68	1.655	4[10]	3.95
$C_{2,q,C_F^3}^{(3)}$	5114	35	938	1.78886×10^6	30394.173	58[289]	0.50924×10^6
$C_{2,q,C_F^3\zeta_3}^{(3)}$	284	8	64	31.02	32.363	7 [18]	27.60
$C_{2,q,C_F^3\zeta_4}^{(3)}$	19	2	5	0.08	0.163	1 [1]	0.47
$C_{2,q,C_F^3\zeta_5}^{(3)}$	19	2	5	0.08	0.163	1 [1]	0.47
$C_{2,q,C_F^2C_A}^{(3)}$	5059	35	930	1.69267×10^6	30122.380	60 [290]	0.47780×10^6
$C_{2,q,C_F^2C_A\zeta_3}^{(3)}$	284	8	64	34.00	33.400	7 [18]	28.53
$C_{2,q,C_F^2C_A\zeta_4}^{(3)}$	48	3	11	0.32	0.643	1[1]	1.01
$C_{2,q,C_F^2C_A\zeta_5}^{(3)}$	19	2	5	0.08	0.167	1 [1]	0.42
$C_{2,q,C_FC_A^2}^{(3)}$	4564	33	863	1.38918×10^6	24567.518	60 [258]	0.34941×10^6
$C_{2,q,C_FC_A^2\zeta_3}^{(3)}$	284	8	63	26.83	29.918	7 [17]	30.46
$C_{2,q,C_FC_A^2\zeta_4}^{(3)}$	48	3	11	0.32	0.643	1 [1]	1.01
$C_{2,q,C_FC_A^2\zeta_5}^{(3)}$	19	2	5	0.08	0.175	1 [1]	0.42
$C_{2,q,C_F^2N_F}^{(3)}$	1762	20	348	40237.45	2339.516	29 [107]	7548.56
$C_{2,q,C_F^2N_F\zeta_3}^{(3)}$	87	4	21	1.94	2.354	3 [5]	2.83
$C_{2,q,C_F^2N_F\zeta_4}^{(3)}$	15	2	3	0.07	0.101	1 [1]	0.34
$C_{2,q,C_FC_AN_F}^{(3)}$	1847	20	360	47661.64	2507.362	29 [111]	7525.89
$C_{2,q,C_FC_AN_F\zeta_3}^{(3)}$	89	4	24	2.47	2.935	3 [8]	3.19
$C_{2,q,C_FC_AN_F\zeta_4}^{(3)}$	15	2	3	0.06	0.101	1 [1]	0.34
$C_{2,q,C_FN_F^2}^{(3)}$	131	5	30	58.00	5.347	7 [22]	8.97
$C_{2,q,C_FN_F^2\zeta_3}^{(3)}$	15	2	3	0.06	0.101	1 [1]	0.38
$C_{2,q,dabc}^{(3)}$	1199	14	242	6583.27	738.498	14 [62]	841.24
$C_{2,q,dabc\zeta_3}^{(3)}$	109	4	25	2.33	3.164	2[7]	2.40
$C_{2,q,dabc\zeta_5}^{(3)}$	8	1	2	0.03	0.041	0[0]	0.10

A complicated example

$C_{2,q} \propto C_F^3$:

- 5114 moments needed. Use a clever way to calculate the input.
- Largest moment: fraction: numerator 13388 digits; denominator 13381 digits.
- CPU time to determine the recurrence: 20.7 days.

- modular prediction of the dimension: 4 h; modular LEQ's: 5.8 days; modular operator GCDs: 11 days; Chinese Remainder + Rat. Reconstruction: 3.8 days. 140 large primes needed.

31 MB recurrence is established; largest integer: 1227 digits; order: 35; degree: 938

- Solved by `sigma` within about one week.
- 3 loop anomalous dimensions: much smaller recurrences & shorter computation times.

$\Longrightarrow$ In practice no method does yet exists to calculate such a high number of moments.

$\Longrightarrow$ Existence proof of a quite general and powerful automatic difference-equation solver, standing rather demanding tests.

Structure of the Results

- We carry out all algebraic reductions, J.B. 2003.
- Different color factor contributions lead to the same or nearly the same amount of sums at a given quantity.
- This points to the fact that the amount of harmonic sums is governed by topology rather than the fields and color.
- The linear harmonic sum representations by Vermaseren et al. 2004/05 require many more sums than our representation.
- There are reductions in the number of sums as $264 \longrightarrow 29$.
- Further use of structural relations will lead to maximally 35 sums for the 3-loop Wilson coefficients; J.B. arxiv: 0901.0837, arXiv:0901.3106.

$$
\begin{aligned}
P^0_{qq}(n) &= C_F\left[4S_1 - \frac{3n^2+3n+2}{n(n+1)}\right] \\
P^{1,-}_{qq}(n) &= C_F^2\bigg[-\frac{3n^6+9n^5+9n^4-5n^3-24n^2-32n-24}{2n^3(n+1)^3} - 16S_{-3} \\
&\quad +S_{-2}\left(\frac{16}{n(n+1)} - 32S_1\right) + S_1\left(\frac{8(2n+1)}{n^2(n+1)^2} - 16S_2\right) + \frac{4\left(3n^2+3n+2\right)}{n(n+1)}S_2 \\
&\quad -16S_3 + 32S_{-2,1} + \frac{16(-1)^n}{(n+1)^3}\bigg] \\
&+ C_AC_F\bigg[-\frac{51n^5+102n^4+655n^3+484n^2+12n+144}{18n^3(n+1)^2} + 8S_{-3} + \frac{268}{9}S_1 \\
&\quad +S_{-2}\left(16S_1 - \frac{8}{n(n+1)}\right) - \frac{44}{3}S_2 + 8S_3 - 16S_{-2,1} - \frac{8(-1)^n}{(n+1)^3}\bigg] \\
&+ C_FN_F\left[\frac{3n^4+6n^3+47n^2+20n-12}{9n^2(n+1)^2} - \frac{40}{9}S_1 + \frac{8}{3}S_2\right] \\
P^{1,+}_{qq}(n) &= C_F^2\bigg[-\frac{3n^6+9n^5+9n^4+59n^3+40n^2+32n+8}{2n^3(n+1)^3} - 16S_{-3} \\
&\quad +S_{-2}\left(\frac{16}{n(n+1)} - 32S_1\right) + S_1\left(\frac{8(2n+1)}{n^2(n+1)^2} - 16S_2\right) \\
&\quad +\frac{4\left(3n^2+3n+2\right)}{n(n+1)}S_2 - 16S_3 + 32S_{-2,1} + \frac{16(-1)^n}{(n+1)^3}\bigg] \\
&+ C_AC_F\bigg[-\frac{51n^5+153n^4+757n^3+851n^2+208n-132}{18n^2(n+1)^3} + 8S_{-3} + \frac{268}{9}S_1 \\
&\quad +S_{-2}\left(16S_1 - \frac{8}{n(n+1)}\right) - \frac{44}{3}S_2 + 8S_3 - 16S_{-2,1} - \frac{8(-1)^n}{(n+1)^3}\bigg] \\
&+ C_FN_F\left[\frac{3n^4+6n^3+47n^2+20n-12}{9n^2(n+1)^2} - \frac{40}{9}S_1 + \frac{8}{3}S_2\right] \\
P^{2,-}_{qq}(n) &= C_F^3\bigg\{\left(\frac{64}{n(n+1)} - 128S_1\right)S_{-2}^2 + \bigg(\frac{16\left(3n^6+9n^5+9n^4+17n^3+6n^2+8n+2\right)}{n^3(n+1)^3} \\
&+ S_1\left(\frac{64\left(3n^2-n+1\right)}{n^2(n+1)^2} - 1408S_2\right) - \frac{64\left(3n^2+3n-11\right)S_2}{n(n+1)} + 1536S_3 + 128S_{-2,1} \\
&- 2304S_{2,1}\bigg)S_{-2} - \frac{16\left(3n^2+3n+2\right)S_2^2}{n(n+1)} - \frac{P_1(n)}{2n^5(n+1)^5} - 576S_{-5}
\end{aligned}
$$

$$
\begin{aligned}
&+ S_{-4}\left(-\frac{16\left(9n^2+9n-26\right)}{n(n+1)} - 832S_1\right) \\
&+ S_{-3}\left(640S_1^2 - \frac{32\left(3n^2+3n+20\right)S_1}{n(n+1)} + \frac{16\left(21n^2+17n+20\right)}{n^2(n+1)^2} - 320S_{-2} - 2240S_2\right) \\
&+ (-1)^n\left(-\frac{48\left(2n^2-n+1\right)}{(n+1)^5} + \frac{128S_{-2}}{(n+1)^3} + \frac{96(5n+3)S_1}{(n+1)^4} - \frac{64S_2}{(n+1)^3}\right) \\
&+ \frac{4\left(13n^4+26n^3+13n^2-16n-20\right)S_3}{n^2(n+1)^2} - \frac{16\left(15n^2+15n+2\right)S_4}{n(n+1)} - 192S_5 - 832S_{-4,1} \\
&+ \frac{896S_{-3,1}}{n(n+1)} + 1152S_{-3,2} + S_1^2\left(-\frac{32\left(3n^2+3n+1\right)}{n^3(n+1)^3} - 768S_{-2,1}\right) - \frac{32\left(15n^2+11n+16\right)S_{-2,1}}{n^2(n+1)^2} \\
&+ S_2\left(\frac{2\left(3n^6+9n^5+9n^4+19n^3+12n^2-4n-16\right)}{n^3(n+1)^3} + 64S_3 + 2176S_{-2,1}\right) \\
&+ \frac{32\left(3n^2+3n-26\right)S_{2,-2}}{n(n+1)} - 1472S_{3,-2} + \frac{64\left(3n^2+3n-2\right)S_{3,1}}{n(n+1)} + 192S_{3,2} + 192S_{4,1} \\
&+ 2304S_{-3,1,1} + 512S_{-2,1,-2} + \frac{384\left(n^2+n-4\right)S_{-2,1,1}}{n(n+1)} + S_1\bigg(64S_2^2 - \frac{64(2n+1)S_2}{n^2(n+1)^2} \\
&+ \frac{4\left(22n^6+186n^5+167n^4-40n^3-115n^2-120n-44\right)}{n^4(n+1)^4} - 192S_3 + 64S_4 - 1792S_{-3,1} \\
&- \frac{192\left(n^2+n-4\right)S_{-2,1}}{n(n+1)} + 1664S_{2,-2} + 256S_{3,1} + 3072S_{-2,1,1}\bigg) + 2304S_{-2,2,1} + 2304S_{2,1,-2} \\
&- 384S_{3,1,1} - 4608S_{-2,1,1,1} \\
&+ \left(C_F^3 - \frac{3}{2}C_F^2C_A\right)\zeta_3\left[-\frac{24\left(5n^4+10n^3+9n^2+4n+4\right)}{n^2(n+1)^2} - 192S_{-2}\right]\bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+ C_AC_F^2\bigg\{\left(256S_1 - \frac{16\left(3n^2+3n+8\right)}{n(n+1)}\right)S_{-2}^2 \\
&+ \bigg[-\frac{8\left(81n^6+243n^5-229n^4-389n^3-130n^2+228n+72\right)}{9n^3(n+1)^3} + \frac{32\left(31n^2+31n-81\right)S_2}{3n(n+1)} \\
&+ S_1\left(1728S_2 - \frac{32\left(134n^4+268n^3+215n^2+45n+54\right)}{9n^2(n+1)^2}\right) - 1792S_3 - 192S_{-2,1} + 2688S_{2,1}\bigg]S_{-2} \\
&+ \frac{176}{3}S_2^2 - \frac{P_2(n)}{36n^5(n+1)^5} + 672S_{-5} + S_{-4}\left(\frac{8\left(97n^2+97n-210\right)}{3n(n+1)} + 1120S_1\right) \\
&+ S_{-3}\bigg(-576S_1^2 + \frac{16\left(31n^2+31n+108\right)S_1}{3n(n+1)} - \frac{8\left(268n^4+536n^3+811n^2+507n+450\right)}{9n^2(n+1)^2} \\
&+ 480S_{-2} + 2656S_2\bigg) + (-1)^n\bigg(\frac{8\left(382n^2+41n-161\right)}{9(n+1)^5} - \frac{256S_{-2}}{(n+1)^3} - \frac{16(127n+121)S_1}{3(n+1)^4} \\
&+ \frac{32S_2}{(n+1)^3}\bigg) - \frac{8\left(385n^4+770n^3+427n^2+6n-126\right)S_3}{9n^2(n+1)^2} + \frac{8\left(151n^2+151n-30\right)S_4}{3n(n+1)} \\
&+ 384S_5 + 864S_{-4,1} - \frac{960S_{-3,1}}{n(n+1)} - 1344S_{-3,2}
\end{aligned}
$$

$$+\ S_2\left(\frac{2\left(453n^5+906n^4+1325n^3+488n^2-120n+144\right)}{9n^3(n+1)^2}-32S_3-2624S_{-2,1}\right)$$
$$+\ \frac{16\left(268n^4+536n^3+625n^2+321n+414\right)S_{-2,1}}{9n^2(n+1)^2}+S_1^2(128S_3+896S_{-2,1})$$
$$-\ \frac{16\left(31n^2+31n-174\right)S_{2,-2}}{3n(n+1)}+1824S_{3,-2}-\frac{32\left(29n^2+29n-24\right)S_{3,1}}{3n(n+1)}-384S_{3,2}-384S_{4,1}$$
$$-\ 2688S_{-3,1,1}-768S_{-2,1,-2}+S_1\left(-\frac{8\left(135n^6+731n^5+245n^4-617n^3-395n^2-309n-144\right)}{9n^4(n+1)^4}\right.$$
$$-\ \frac{2144}{9}S_2+\frac{32\left(31n^2+31n-12\right)S_3}{3n(n+1)}+160S_4+1920S_{-3,1}+\frac{32\left(31n^2+31n-84\right)S_{-2,1}}{3n(n+1)}$$
$$-\ \left.1856S_{2,-2}-512S_{3,1}-3584S_{-2,1,1}\right)-\frac{64\left(31n^2+31n-84\right)S_{-2,1,1}}{3n(n+1)}-2688S_{-2,2,1}-2688S_{2,1,-2}$$
$$+\ \left.768S_{3,1,1}+5376S_{-2,1,1,1}\right\}$$
$$+\ C_A^2C_F\left[\left(\frac{24\left(n^2+n+2\right)}{n(n+1)}-96S_1\right)S_{-2}^2+\left(\frac{8\left(27n^6+81n^5-155n^4-271n^3-92n^2+78n+27\right)}{9n^3(n+1)^3}\right.\right.$$
$$+\ S_1\left(\frac{16\left(134n^4+268n^3+188n^2+54n+45\right)}{9n^2(n+1)^2}-512S_2\right)-\frac{32\left(11n^2+11n-24\right)S_2}{3n(n+1)}+512S_3$$
$$+\ \left.64S_{-2,1}-768S_{2,1}\right)S_{-2}+\frac{P_3(n)}{108n^5(n+1)^5}-192S_{-5}+S_{-4}\left(-\frac{8\left(35n^2+35n-66\right)}{3n(n+1)}-352S_1\right)$$
$$+\ (-1)^n\left(-\frac{16\left(82n^2+17n-47\right)}{9(n+1)^5}+\frac{96S_{-2}}{(n+1)^3}+\frac{16(41n+47)S_1}{3(n+1)^4}\right)$$
$$+\ S_{-3}\left(128S_1^2-\frac{16\left(11n^2+11n+24\right)S_1}{3n(n+1)}+\frac{8\left(134n^4+268n^3+311n^2+177n+135\right)}{9n^2(n+1)^2}\right.$$
$$-\ \left.160S_{-2}-768S_2\right)+\frac{4\left(389n^4+778n^3+398n^2+9n-81\right)S_3}{9n^2(n+1)^2}-\frac{8\left(55n^2+55n-24\right)S_4}{3n(n+1)}$$
$$-\ 160S_5-224S_{-4,1}+\frac{256S_{-3,1}}{n(n+1)}+384S_{-3,2}+S_1^2(-64S_3-256S_{-2,1})$$
$$-\ \frac{16\left(134n^4+268n^3+245n^2+111n+135\right)S_{-2,1}}{9n^2(n+1)^2}+S_2\left(768S_{-2,1}-\frac{4172}{27}\right)$$
$$+\ \frac{16\left(11n^2+11n-48\right)S_{2,-2}}{3n(n+1)}-544S_{3,-2}+\frac{32\left(11n^2+11n-12\right)S_{3,1}}{3n(n+1)}$$
$$+\ 192S_{3,2}+192S_{4,1}+768S_{-3,1,1}+256S_{-2,1,-2}+\frac{64\left(11n^2+11n-24\right)S_{-2,1,1}}{3n(n+1)}$$
$$+\ S_1\left(\frac{2\left(245n^8+980n^7+1542n^6+1524n^5+851n^4+100n^3+36n^2+22n-6\right)}{3n^4(n+1)^4}\right.$$
$$-\ \frac{8\left(11n^2+11n-8\right)S_3}{n(n+1)}-128S_4-512S_{-3,1}-\frac{32\left(11n^2+11n-24\right)S_{-2,1}}{3n(n+1)}$$
$$+\ \left.512S_{2,-2}+256S_{3,1}+1024S_{-2,1,1}\right)+768S_{-2,2,1}+768S_{2,1,-2,}-384S_{3,1,1}$$
$$-\ \left.1536S_{-2,1,1,1}\right]$$
$$+\ C_A^2C_F\zeta_3\left[-\frac{12\left(5n^4+10n^3+9n^2-4n-4\right)}{n^2(n+1)^2}-96S_{-2}\right]$$
$$+\ C_FN_F^2\left[\frac{51n^6+153n^5+57n^4+35n^3+96n^2+16n-24}{27n^3(n+1)^3}-\frac{16}{27}S_1-\frac{80}{27}S_2+\frac{16}{9}S_3\right]$$
$$+\ C_F^2N_F\left[-\frac{32}{3}S_2^2-\frac{4\left(15n^4+30n^3+79n^2+16n-24\right)S_2}{9n^2(n+1)^2}\right.$$
$$+\ \frac{207n^8+828n^7+1443n^6+1123n^5-38n^4-779n^3-632n^2+120}{9n^4(n+1)^4}-\frac{128}{3}S_{-4}$$
$$+\ S_{-3}\left(\frac{32\left(10n^2+10n+3\right)}{9n(n+1)}-\frac{64}{3}S_1\right)+(-1)^n\left(\frac{64S_1}{3(n+1)^3}-\frac{128(4n+1)}{9(n+1)^4}\right)$$
$$+\ S_{-2}\left(-\frac{32\left(16n^2+10n-3\right)}{9n^2(n+1)^2}+\frac{640}{9}S_1-\frac{128}{3}S_2\right)+\frac{16\left(29n^2+29n+12\right)S_3}{9n(n+1)}-\frac{128}{3}S_4$$
$$+\ S_1\left(-\frac{2\left(165n^5+330n^4+165n^3+160n^2-16n-96\right)}{9n^3(n+1)^2}+\frac{320}{9}S_2-\frac{128}{3}S_3-\frac{128}{3}S_{-2,1}\right)$$
$$-\ \left.\frac{64\left(10n^2+10n-3\right)S_{-2,1}}{9n(n+1)}+\frac{64}{3}S_{2,-2}+\frac{64}{3}S_{3,1}+\frac{256}{3}S_{-2,1,1}\right]$$
$$+\ \left(C_F^2-C_FC_A\right)N_F\zeta_3\left[32S_1-\frac{8\left(3n^2+3n+2\right)}{n(n+1)}\right]$$
$$+\ C_AC_FN_F\left[-\frac{2\left(270\ n^7+810n^6-463n^5-1392n^4-211n^3-206n^2-156n+144\right)}{27n^4(n+1)^3}\right.$$
$$+\ \frac{64}{3}S_{-4}+S_{-3}\left(\frac{32}{3}S_1-\frac{16\left(10n^2+10n+3\right)}{9n(n+1)}\right)+(-1)^n\left(\frac{64(4n+1)}{9(n+1)^4}-\frac{32S_1}{3(n+1)^3}\right)$$
$$+\ \frac{1336}{27}S_2+S_{-2}\left(\frac{16\left(16n^2+10n-3\right)}{9n^2(n+1)^2}-\frac{320}{9}S_1+\frac{64}{3}S_2\right)-\frac{8\left(14n^2+14n+3\right)S_3}{3n(n+1)}+\frac{80}{3}S_4$$
$$+\ \frac{32\left(10n^2+10n-3\right)S_{-2,1}}{9n(n+1)}+S_1\left(-\frac{4\left(209n^6+627n^5+627n^4+281n^3+36n^2+36n+18\right)}{27n^3(n+1)^3}\right.$$
$$+\ \left.\left.16S_3+\frac{64}{3}S_{-2,1}\right)-\frac{32}{3}S_{2,-2}-\frac{64}{3}S_{3,1}-\frac{128}{3}S_{-2,1,1}\right]$$

$$P_{qq}^{2,+}\ =\ C_F^3\left[\left(\frac{64}{n(n+1)}-128S_1\right)S_{-2}^2+\left(\frac{16\left(3n^6+9n^5+9n^4+n^3+2n^2+4n+2\right)}{n^3(n+1)^3}\right.\right.$$
$$+\ S_1\left(-\frac{64\left(3n^2+7n+5\right)}{n^2(n+1)^2}-1408S_2\right)-\frac{64\left(3n^2+3n-11\right)S_2}{n(n+1)}+1536S_3+128S_{-2,1}$$

$$- \quad 2304S_{2,1}\Big)S_{-2} - \frac{16\,(3n^2+3n+2)\,S_2^2}{n(n+1)} - \frac{P_4(n)}{2n^5(n+1)^5} - 576S_{-5}$$

$$+ \quad S_{-4}\left(-\frac{16\,(9n^2+9n-26)}{n(n+1)} - 832S_1\right) + S_{-3}\Bigg(640S_1^2 - \frac{32\,(3n^2+3n+20)\,S_1}{n(n+1)}$$

$$+ \quad \frac{16\,(9n^2+5n+8)}{n^2(n+1)^2} - 320S_{-2} - 2240S_2\Bigg) + (-1)^n\Bigg(\frac{16\,(2n^2+11n+1)}{(n+1)^5} + \frac{128S_{-2}}{(n+1)^3}$$

$$+ \quad \frac{96(5n+3)S_1}{(n+1)^4} - \frac{64S_2}{(n+1)^3}\Bigg) + \frac{4\,(13n^4+26n^3+13n^2-16n-20)\,S_3}{n^2(n+1)^2}$$

$$- \quad \frac{16\,(15n^2+15n+2)\,S_4}{n(n+1)} - 192S_5 - 832S_{-4,1} + \frac{896S_{-3,1}}{n(n+1)} + 1152S_{-3,2}$$

$$+ \quad S_1^2\left(-\frac{32\,(3n^2+3n+1)}{n^3(n+1)^3} - 768S_{-2,1}\right) - \frac{32\,(3n^2-n+4)\,S_{-2,1}}{n^2(n+1)^2}$$

$$+ \quad S_2\left(\frac{2\,(3n^6+9n^5+9n^4+83n^3+76n^2+60n+16)}{n^3(n+1)^3} + 64S_3 + 2176S_{-2,1}\right)$$

$$+ \quad \frac{32\,(3n^2+3n-26)\,S_{2,-2}}{n(n+1)} - 1472S_{3,-2} + \frac{64\,(3n^2+3n-2)\,S_{3,1}}{n(n+1)} + 192S_{3,2} + 192S_{4,1}$$

$$+ \quad 2304S_{-3,1,1} + 512S_{-2,1,-2} + \frac{384\,(n^2+n-4)\,S_{-2,1,1}}{n(n+1)} + S_1\Bigg(64S_2^2 - \frac{64(2n+1)S_2}{n^2(n+1)^2}$$

$$+ \quad \frac{4\,(22n^6-54n^5+23n^4+88n^3+197n^2+160n+52)}{n^4(n+1)^4} - 192S_3 + 64S_4 - 1792S_{-3,1}$$

$$- \quad \frac{192\,(n^2+n-4)\,S_{-2,1}}{n(n+1)} + 1664S_{2,-2} + 256S_{3,1} + 3072S_{-2,1,1}\Bigg) + 2304S_{-2,2,1}$$

$$+ \quad 2304S_{2,1,-2} - 384S_{3,1,1} - 4608S_{-2,1,1,1}\Bigg]$$

$$+ \quad C_F^3\zeta_3\left[-\frac{24\,(5n^4+10n^3+n^2-4n-4)}{n^2(n+1)^2} - 192S_{-2}\right]$$

$$+ \quad C_AC_F^2\Bigg\{\left(256S_1 - \frac{16\,(3n^2+3n+8)}{n(n+1)}\right)S_{-2}^2$$

$$+ \quad \Bigg(-\frac{8\,(81n^5+243n^4-337n^3-1181n^2-526n-60)}{9n^2(n+1)^3} + \frac{32\,(31n^2+31n-81)\,S_2}{3n(n+1)}$$

$$+ \quad S_1\left(1728S_2 - \frac{32\,(134n^4+268n^3+89n^2-81n-72)}{9n^2(n+1)^2}\right) - 1792S_3 - 192S_{-2,1} + 2688S_{2,1}\Bigg)S_{-2}$$

$$+ \quad \frac{176}{3}S_2^2 - \frac{P_5(n)}{36n^4(n+1)^4} + 672S_{-5} + S_{-4}\left(\frac{8\,(97n^2+97n-210)}{3n(n+1)} + 1120S_1\right)$$

$$+ \quad S_{-3}\Bigg(-576S_1^2 + \frac{16\,(31n^2+31n+108)\,S_1}{3n(n+1)} - \frac{8\,(268n^4+536n^3+487n^2+183n+126)}{9n^2(n+1)^2}$$

$$+ \quad 480S_{-2} + 2656S_2\Bigg) + (-1)^n\left(\frac{8(346n-125)}{9(n+1)^4} - \frac{256S_{-2}}{(n+1)^3} - \frac{16(103n+73)S_1}{3(n+1)^4} + \frac{32S_2}{(n+1)^3}\right)$$

$$- \quad \frac{8\,(385n^4+770n^3+427n^2+6n-126)\,S_3}{9n^2(n+1)^2} + \frac{8\,(151n^2+151n-30)\,S_4}{3n(n+1)} + 384S_5$$

$$+ \quad 864S_{-4,1} - \frac{960S_{-3,1}}{n(n+1)} - 1344S_{-3,2} + S_2\Bigg(\frac{2\,(453n^5+1359n^4+2231n^3+1525n^2+80n-264)}{9n^2(n+1)^3}$$

$$- \quad 32S_3 - 2624S_{-2,1}\Bigg) + \frac{16\,(268n^4+536n^3+301n^2-3n+90)\,S_{-2,1}}{9n^2(n+1)^2} + S_1^2(128S_3 + 896S_{-2,1})$$

$$- \quad \frac{16\,(31n^2+31n-174)\,S_{2,-2}}{3n(n+1)} + 1824S_{3,-2} - \frac{32\,(29n^2+29n-24)\,S_{3,1}}{3n(n+1)} - 384S_{3,2} - 384S_{4,1}$$

$$- \quad 2688S_{-3,1,1} - 768S_{-2,1,-2} + S_1\Bigg(-\frac{8\,(135n^6-649n^5-1039n^4-569n^3+487n^2+621n+216)}{9n^4(n+1)^4}$$

$$- \quad \frac{2144}{9}S_2 + \frac{32\,(31n^2+31n-12)\,S_3}{3n(n+1)} + 160S_4 + 1920S_{-3,1} + \frac{32\,(31n^2+31n-84)\,S_{-2,1}}{3n(n+1)}$$

$$- \quad 1856S_{2,-2} - 512S_{3,1} - 3584S_{-2,1,1}\Bigg) - \frac{64\,(31n^2+31n-84)\,S_{-2,1,1,n}}{3n(n+1)} - 2688S_{-2,2,1}$$

$$- \quad 2688S_{2,1,-2} + 768S_{3,1,1} + 5376S_{-2,1,1,1}\Bigg]$$

$$+ \quad C_AC_F^2\zeta_3\left[\frac{36\,(5n^4+10n^3+n^2-4n-4)}{n^2(n+1)^2} + 288S_{-2}\right]$$

$$+ \quad C_A^2C_F\left(\frac{24\,(n^2+n+2)}{n(n+1)} - 96S_1\right)S_{-2}^2 + \Bigg(\frac{8\,(27n^6+81n^5-209n^4-595n^3-272n^2-48n-9)}{9n^3(n+1)^3}$$

$$+ \quad S_1\left(\frac{16\,(134n^4+268n^3+116n^2-18n-27)}{9n^2(n+1)^2} - 512S_2\right) - \frac{32\,(11n^2+11n-24)\,S_2}{3n(n+1)} + 512S_3$$

$$+ \quad 64S_{-2,1} - 768S_{2,1}\Bigg)S_{-2} + \frac{P_6(N)}{108n^3(n+1)^5} - 192S_{-5} + S_{-4}\left(-\frac{8\,(35n^2+35n-66)}{3n(n+1)} - 352S_1\right)$$

$$+ \quad (-1)^n\left(-\frac{16\,(91n^2+80n-29)}{9(n+1)^5} + \frac{96S_{-2}}{(n+1)^3} + \frac{16(29n+23)S_1}{3(n+1)^4}\right)$$

$$+ \quad S_{-3}\Bigg(128S_1^2 - \frac{16\,(11n^2+11n+24)\,S_1}{3n(n+1)} + \frac{8\,(134n^4+268n^3+203n^2+69n+27)}{9n^2(n+1)^2}$$

$$- \quad 160S_{-2} - 768S_2\Bigg) + \frac{4\,(389n^4+778n^3+398n^2+9n-81)\,S_3}{9n^2(n+1)^2} - \frac{8\,(55n^2+55n-24)\,S_4}{3n(n+1)}$$

$$- \quad 160S_5 - 224S_{-4,1} + \frac{256S_{-3,1}}{n(n+1)} + 384S_{-3,2} + S_1^2(-64S_3 - 256S_{-2,1})$$

$$- \quad \frac{16\,(134n^4+268n^3+137n^2+3n+27)\,S_{-2,1}}{9n^2(n+1)^2} + S_2\left(768S_{-2,1} - \frac{4172}{27}\right)$$

$$+ \quad \frac{16\,(11n^2+11n-48)\,S_{2,-2}}{3n(n+1)} - 544S_{3,-2} + \frac{32\,(11n^2+11n-12)\,S_{3,1}}{3n(n+1)} + 192S_{3,2}$$

$$
\begin{aligned}
&+\; 192S_{4,1}+768S_{-3,1,1}+256S_{-2,1,-2}+\frac{64\,(11n^2+11n-24)\,S_{-2,1,1}}{3n(n+1)}\\
&+\; S_1\Bigg(\frac{2\,(245n^8+980n^7+1542n^6+964n^5+211n^4-60n^3+156n^2+222n+90)}{3n^4(n+1)^4}\\
&-\; \frac{8\,(11n^2+11n-8)\,S_3}{n(n+1)}-128S_4-512S_{-3,1}\\
&-\; \frac{32\,(11n^2+11n-24)\,S_{-2,1}}{3n(n+1)}+512S_{2,-2}+256S_{3,1}+1024S_{-2,1,1}\Bigg)+768S_{-2,2,1}\\
&+\; 768S_{2,1,-2}-384S_{3,1,1}-1536S_{-2,1,1,1}\Bigg\}\\
&+\; C_A^2C_F\zeta_3\left[-\frac{12\,(5n^4+10n^3+n^2-4n-4)}{n^2(n+1)^2}-96S_{-2}\right]\\
&+\; C_F^2N_F\Bigg\{-\frac{32}{3}S_2^2-\frac{4\,(15n^4+30n^3+79n^2+16n-24)\,S_2}{9n^2(n+1)^2}+\frac{P_7(n)}{9n^4(n+1)^4}-\frac{128}{3}S_{-4}\\
&+\; S_{-3}\left(\frac{32\,(10n^2+10n+3)}{9n(n+1)}-\frac{64}{3}S_1\right)+(-1)^n\left(\frac{64S_1}{3(n+1)^3}-\frac{128(4n+1)}{9(n+1)^4}\right)\\
&+\; S_{-2}\left(-\frac{32\,(16n^2+10n-3)}{9n^2(n+1)^2}+\frac{640}{9}S_1-\frac{128}{3}S_2\right)+\frac{16\,(29n^2+29n+12)\,S_3}{9n(n+1)}-\frac{128}{3}S_4\\
&+\; S_1\left(-\frac{2\,(165n^5+495n^4+495n^3+517n^2+336n+80)}{9n^2(n+1)^3}+\frac{320}{9}S_2-\frac{128}{3}S_3-\frac{128}{3}S_{-2,1}\right)\\
&-\; \frac{64\,(10n^2+10n-3)\,S_{-2,1}}{9n(n+1)}+\frac{64}{3}S_{2,-2}+\frac{64}{3}S_{3,1}+\frac{256}{3}S_{-2,1,1}\Bigg\}\\
&+\; C_F^2N_F\zeta_3\left[32S_1-\frac{8\,(3n^2+3n+2)}{n(n+1)}\right]\\
&+\; C_FN_F^2\left[\frac{51n^6+153n^5+57n^4+35n^3+96n^2+16n-24}{27n^3(n+1)^3}-\frac{16}{27}S_1-\frac{80}{27}S_2+\frac{16}{9}S_3\right]\\
&+\; C_AC_FN_F\Bigg[-\frac{2\,(270n^7+1080n^6+383n^5-979n^4-571n^3+507n^2+106n-132)}{27n^3(n+1)^4}\\
&+\; \frac{64}{3}S_{-4}+S_{-3}\left(\frac{32}{3}S_1-\frac{16\,(10n^2+10n+3)}{9n(n+1)}\right)+(-1)^n\left(\frac{64(4n+1)}{9(n+1)^4}-\frac{32S_1}{3(n+1)^3}\right)\\
&+\; \frac{1336}{27}S_2+S_{-2}\left(\frac{16\,(16n^2+10n-3)}{9n^2(n+1)^2}-\frac{320}{9}S_1+\frac{64}{3}S_2\right)-\frac{8\,(14n^2+14n+3)\,S_3}{3n(n+1)}+\frac{80}{3}S_4\\
&+\; \frac{32\,(10n^2+10n-3)\,S_{-2,1}}{9n(n+1)}+S_1\Bigg(-\frac{4\,(209n^6+627n^5+627n^4+137n^3-108n^2-108n-54)}{27n^3(n+1)^3}\\
&+\; 16S_3+\frac{64}{3}S_{-2,1}\Bigg)-\frac{32}{3}S_{2,-2}-\frac{64}{3}S_3,1-\frac{128}{3}S_{-2,1,1}\Bigg]\\
&+\; C_AC_FN_F\zeta_3\left[\frac{8\,(3n^2+3n+2)}{n(n+1)}-32S_1\right]
\end{aligned}
$$

$$
\begin{aligned}
P_{qq}^{2,-,dabc} &= \frac{d_{abc}d^{abc}}{N_c}N_F\Bigg[-\frac{P_8(n)}{3n^5(n+1)^5(n+2)^3}+\frac{4\,(n^2+n+2)\,S_{-3}}{n^2(n+1)^2}-\frac{P_9(n)S_1}{3n^4(n+1)^4(n+2)^3}\\
&+\; S_{-2}\left(-\frac{8S_1\,(n^2+n+2)^2}{(n-1)n^2(n+1)^2(n+2)}-\frac{4\,(n^6+3n^5-8n^4-21n^3-23n^2-12n-4)}{(n-1)n^3(n+1)^3(n+2)}\right)\\
&+\; (-1)^n\Bigg(\frac{16\,(5n^6+29n^5+78n^4+118n^3+114n^2+72n+16)\,S_1}{3(n-1)n^2(n+1)^3(n+2)^3}\\
&-\; \frac{4\,(13n^8+74n^7+179n^6+314n^5+644n^4+1000n^3+816n^2+352n+64)}{3(n-1)n^3(n+1)^4(n+2)^3}\Bigg)\\
&-\; \frac{2\,(n^2+n+2)\,S_3}{n^2(n+1)^2}-\frac{8\,(n^2+n+2)\,S_{-2,1}}{n^2(n+1)^2}\Bigg]
\end{aligned}
$$

Other Processes

- The present method can be applied irrespectively of the loop order to all single scale processes.
- As has been found before J.B. & Ravindran 2004/05, J.B. & Moch 2005, J.B. & S. Klein 2007 representing a large number of 2- and 3-loop processes in terms of harmonic sums, the basis elements emerging are always the same.

  {anomalous dimensions, Wilson coefficients, space- and time-like, polarized/unpolarized, Drell-Yan process, hadronic Higgs Boson production in the heavy mass limit, HO QED corrections in e^+e^- annihilation, soft+virtual corrections to Bhabha scattering}.
- The formalism also applies to Heavy Flavor Wilson Coefficients at $Q^2 \gg m^2$, c.f. Bierenbaum, J.B., Klein 2007/08; arxiv:0904.3563 [hep-ph], DESY 09–057.
- Basis to $w = 6$, c.f. J.B., arxiv 0901.0837.

5. Conclusions

- We established a general algorithm to calculate the exact expression for single scale quantities from a finite (suitably large) number of moments (zero scale quantities).
- The latter ones are much more easily calculable.
- We applied the method to the anomalous dimensions and Wilson coefficients up to 3-loop order.
- To solve 3-loop problems this way is not possible at present, since the number of required moments is too large for the methods available.
- We attempted to solve the quantities for all color projections at once. This problem is too voluminous.
- Yet we showed that giant difference equations [order 35; degree $\sim$ 1000] can be reliably and fast established and solved unconditionally for advanced problems in Quantum Field Theory.