Analytic Integration Methods in QFT: an Introduction

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DESY

- Introduction
- Principle Computation Steps
- Important Function Spaces
- The Different Calculation Methods
- A Reconstruction Method in the post-Minkowskian Approach
- Aspects of elliptic solutions
- Conclusions

Loops and Legs:

Feynman diagrams describe elementary scattering processes between bosons and fermions in Quantum Field Theory (QFT). Here we will thoroughly refer to renormalizable QFTs or effective field theories.

Where are these techniques important?

1. Perturbation Theory of the Standard Model and its renormalizable extensions.
2. String amplitude calculations
3. Perturbative calculations in Gravity
4. non-relativistic field theories in vacuum and at finite temperature and/or density

We will calculate Feynman diagrams. These are skeletons according to Feynman rules, connecting vertices with propagators.

They possess external lines: The Legs.

They possess internal closed lines: The Loops.

The machine, for which we perform the calculations:

LHC, Geneva/CH

Why are these calculations important ?

1. Precision extraction of coupling constants: $\alpha_s(M_Z)$@1%
2. Do couplings unite at high scales and in which field theories?
3. Precision measurements of m_c, m_b, m_t at LHC and a future ILC
4. Precision understanding of the Higgs and top sector (at the LHC, ILC and possibly other machines)
5. Unravel the mathematical structure of microscopic processes analytically: get further with the Stueckelberg-Feynman programme as far as you can.

$\Longrightarrow$ Genetic Code of the Micro Cosmos

The Computer Algebra Landscape in Quantum Field Theory

Many symbolic systems and packages written using various languages are in use and will be in use in the future.

1. Fortran, C
2. Mathematica
3. Maple
4. FORM
5. GiNac
6. Sage
7. Pari, and others

- Many calculations bind different packages by shell-scripts to a general computer-algebraic work-flow to solve large-scale problems.
- Condition: the average time used in the parts is not tiny.
- Allows for natural checkpoints; in- and output pattern has to be provided in an automated form.

Our computer-algebra cluster currently consists of 13 units with $\sim$ 7 Tbyte RAM and $\sim$ 200 Tbyte fast disc; we use hundreds of Mathematica licenses, Thanks to RISC at JKU Linz!

Principle Computation Steps

1. Generate the Feynman diagrams: $O(100 - \text{sev.}\ 10^6)$ package QGRAF [Fortran] [P. Nogueira]
2. Calculate all group theoretic structures: package COLOR [FORM] [T. van Ritbergen et al.]
3. Perform all tensor and Dirac-matrix calculations in $4 + \varepsilon$ dimensions, perform all radial momentum integrals: package FORM; [J. Vermaseren et al.]
   remaining: Feynman parameter integrals.
4. Alternatively: reduce to a small number of scalar master integrals, to be calculated by other methods.
5. All accessible Gauß-Stokes integrals are used to reduce millions of scalar integrals often to $O(100 - 5000)$ master integrals; different codes. Examples: [S. Laporta, Anastasiou, Studerus/Manteuffel: Reduze2, Marquard, Lee, and many more.]
   $\Longrightarrow$ P.Marquard, J. Vermaseren, H. Frellesvig

Principle Computation Steps

Example: 3-loop heavy flavor corrections to DIS
[S. Wolfram computed the 1-loop correction in 1978, after E. Witten 1976]

The reduction to master integrals produces 1.6 Tbyte C-output of relations to determine the master integrals.

100.000's of scalar integrals $\Longrightarrow$ 687 3-loop master integrals.

In the calculation of the master integrals Mathematica plays

a key-role.

The calculation methods

1. Generalized Hypergeometric Functions and Extensions $\Longrightarrow$ P. Paule, M. Kamykov
2. Mellin-Barnes techniques
3. PSLQ: zero-dimensional integrals $\Longrightarrow$ D. Broadhurst
4. Guessing: one-dimensional integrals $\Longrightarrow$ this talk
5. Hyperlogarithms $\Longrightarrow$ E. Panzer
6. Solution of master-integrals using difference and differential equations $\Longrightarrow$ C. Schneider, J.-A. Weil, J. Henn
7. Risch algorithms $\Longrightarrow$ C. Raab
8. Summation techniques: construction of difference rings and fields $\Longrightarrow$ C. Schneider
9. Holonomic Integration $\Longrightarrow$ C. Koutschan
10. (multivalued) Almkvist-Zeilberger algorithm $\Longrightarrow$ J. Ablinger
11. Expansion by regions $\Longrightarrow$ V. Smirnov
12. Elliptic integrals and related topics $\Longrightarrow$ S. Weinzierl, J. Brödel
13. Cuttiung techniques $\Longrightarrow$ D. Kreimer
14. Multi-leg applications $\Longrightarrow$ M. von Hippel, G. Papathanasiou, J. Bartels

Nested Sums and Iterative Integrals Function Spaces

Nested sums:

$$S_{1\ldots m}(N) = \sum_{k_1=1}^{N} s(k_1) \sum_{k_2=1}^{k_1} s(k_2) \ldots \sum_{k_m=1}^{k_{m-1}} s(k_m)$$

Iterated integrals:

$$F_{1\ldots l}(x) = \int_0^x dy_1 f_1(y_1) \int_0^{y_1} dy_2 f_2(y_2) \ldots \int_0^{y_{l-1}} dy_l f_l(y_l)$$

Mellin transform:

$$\sum_\alpha c_\alpha S_\alpha(N) = \int_0^1 dx x^{N-1} F(x)$$

$$F_1(x) \cdot F_2(x) = (F_1 ⧢ F_2)(x) = F_{1,2}(x) + F_{2,1}(x)$$ Shuffling

$$S_k(N) \cdot S_l(N) = S_{k,l}(N) + S_{l,k}(N) - S_{k \wedge l}(N), k \wedge l = \mathrm{sign}(k)\mathrm{sign}(l)(|k| + |l|), k, l \in \mathbb{Z} \backslash \{0\}$$ Stuffling

For further details, see: J. Ablinger, JB, C. Schneider, 1304.7071, 1310.5645, JB, C. Schneider, 1809.02889

Function Spaces

Sums

Harmonic Sums

$$\sum_{k=1}^{N} \frac{1}{k} \sum_{l=1}^{k} \frac{(-1)^l}{l^3}$$

gen. Harmonic Sums

$$\sum_{k=1}^{N} \frac{(1/2)^k}{k} \sum_{l=1}^{k} \frac{(-1)^l}{l^3}$$

Cycl. Harmonic Sums

$$\sum_{k=1}^{N} \frac{1}{(2k+1)} \sum_{l=1}^{k} \frac{(-1)^l}{l^3}$$

Binomial Sums

$$\sum_{k=1}^{N} \frac{1}{k^2} \binom{2k}{k} (-1)^k$$

Integrals

Harmonic Polylogarithms

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{1+z}$$

gen. Harmonic Polylogarithms

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{z-3}$$

Cycl. Harmonic Polylogarithms

$$\int_0^x \frac{dy}{1+y^2} \int_0^y \frac{dz}{1-z+z^2}$$

root-valued iterated integrals

$$\int_0^x \frac{dy}{y} \int_0^y \frac{dz}{z\sqrt{1+z}}$$

iterated integrals containing elliptic structures

$$\int_0^z dx \frac{\ln(x)}{1+x} {}_2F_1\left[\begin{matrix} \frac{4}{3}, \frac{5}{3} \\ 2 \end{matrix}; \frac{x^2(x^2-9)^2}{(x^2+3)^3}\right]$$

Special Numbers

multiple zeta values

$$\int_0^1 dx \frac{\mathrm{Li}_3(x)}{1+x} = -2\mathrm{Li}_4(1/2) + \ldots$$

gen. multiple zeta values

$$\int_0^1 dx \frac{\ln(x+2)}{x-3/2} = \mathrm{Li}_2(1/3) + \ldots$$

cycl. multiple zeta values

$$\mathbf{C} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

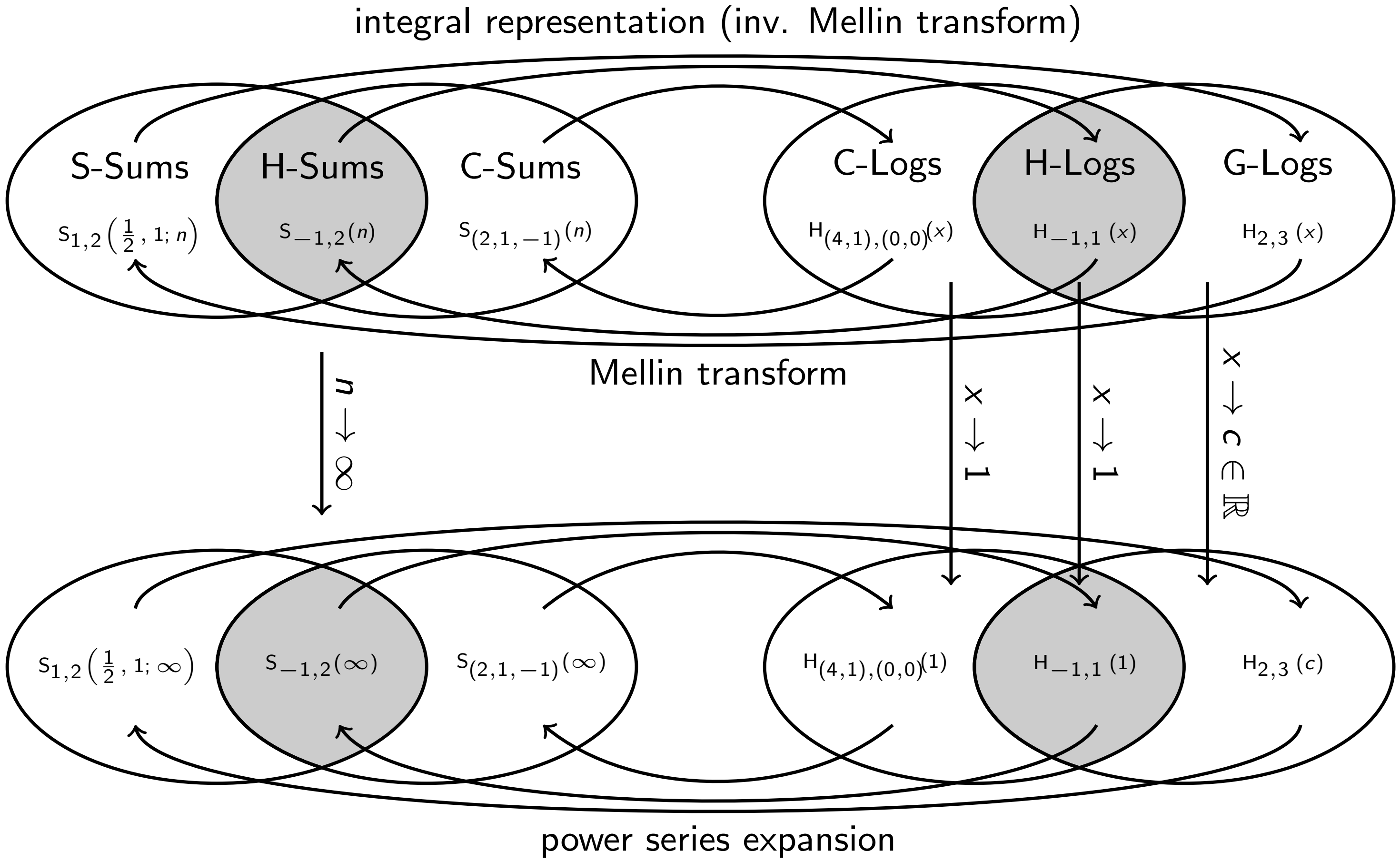
associated numbers

$$\mathrm{H}_{8,w_3} = 2\mathrm{arccot}(\sqrt{7})^2$$

associated numbers

$$\int_0^1 dx \; {}_2F_1\left[\begin{matrix} \frac{4}{3}, \frac{5}{3} \\ 2 \end{matrix}; \frac{x^2(x^2-9)^2}{(x^2+3)^3}\right]$$

shuffle, stuffle, and various structural relations $\Longrightarrow$ algebras



square-root valued letters $\Longleftrightarrow$ nested binomial sums $\binom{2i}{i}$

non-iterative iterative integrals $\Longrightarrow$ Iterate on ${}_2F_1$'s: rat. argument

(special cases: complete elliptic integrals) [arXiv:1706.01299]

Spill-Off: New Function Classes and Algebras

- 1998: Harmonic Sums [Vermaseren; JB]
- 1999: Harmonic Polylogarithms [Remiddi, Vermaseren]
- 2000,2003, 2009: Analytic continuation of harmonic sums, systematic algebraic reduction; structural relations [JB]
- 2001: Generalized Harmonic Sums [Moch, Uwer, Weinzierl]
- 2004: Infinite harmonic (inverse) binomial sums [Davydychev, Kalmykov; Weinzierl]
- 2011: (generalized) Cyclotomic Harmonic Sums, polylogarithms and numbers [Ablinger, JB, Schneider]
- 2013: Systematic Theory of Generalized Harmonic Sums, polylogarithms and numbers [Ablinger, JB, Schneider]
- 2014: Finite nested Generalized Cyclotomic Harmonic Sums with (inverse) Binomial Weights [Ablinger, JB, Raab, Schneider]
- 2014-: Elliptic integrals with (involved) rational arguments.

Particle Physics Generates NEW Mathematics.

Classes of generalized hypergeometric functions

At lower number of legs and/or loops Feynman integrals happen to be represented by these functions.
After suitable mappings these functions have compact representations in infinite (multiple) absolutely convergent sums.
This allows for the Laurent-expansion in ε under the summation operator.

Important Examples:

1. $B(a, b)$
2. ${}_pF_q(a_i; b_j; x)$; always single sums
3. Appell functions; double sums
4. Kampé de Feriet functions, Horn functions and higher; more sums

The sums may be expanded and summed using algorithms like

nestedsums, xsummer, HarmonicSums, Sigma, EvaluateMultiSums

Classes of generalized hypergeometric functions

Example:

Integrals of the following type emerge:

$$
\begin{aligned}
I_1(z) &= \int_0^1 dy y^\delta (1-y)^\eta \int_0^1 dx x^{\beta-1}(1-x)^{\gamma-\beta-1}(1-xyz)^{-\alpha} \\
&= B(\beta,\gamma-\beta) \int_0^1 dy y^\delta (1-y)^\eta \;\; {}_2F_1(\alpha,\beta;\gamma;yz) \\
&= B(\beta,\gamma-\beta) B(\delta,\eta-\delta) \;\; {}_3F_2(\delta,\alpha,\beta;\eta,\gamma;z)
\end{aligned}
$$

All ${}_pF_q$'s have single series representations. One series counts as one integral.

$$
{}_pF_q(a_1,...,a_p;b_1...b_q;z) = \sum_{k=0}^\infty \frac{(a_1)_k...(a_p)_k}{(b_1)_k...(b_q)_k} \frac{z^k}{k!}
$$

Representations in terms of Mellin-Barnes Integrals

Used to resolve sums in denominators. In a way a counterpart to the binomial expansion of numerators.

$$\frac{1}{(A+B)^c} = \oint_{-i\infty+\gamma}^{i\infty+\gamma} d\sigma \frac{\Gamma(-\sigma)\Gamma(\sigma+c)}{\Gamma(c)} A^\sigma B^{-\sigma+c}$$

The contour integral covers the residues and closes either left or right.
One may use Beta-functions and (generalized) hypergeometric functions to perform the integrals further.
Important: One has to be able to undo the Mellin-Barnes integral(s).

1. Barnes Lemmas; map to Γ-functions, (in the most simple cases).
2. Residue theorem $\Longrightarrow$ leads to nested sums $\Longrightarrow$ use SIGMA.

The PSLQ Method

Seek an Integer Relation over a basis of special numbers out of a special class.

Example:

$$I = \int_0^1 dx \frac{\mathrm{Li}_3(x)}{1+x}$$

The integral is of "transcendentality" $\tau = 4$.

The expected HPL(1) basis is spanned by:

$\ln^4(2),\ \ln(2)\zeta_3,\ \ln^2(2)\zeta_2,\ \zeta_2^2,\ \mathrm{Li}_4(1/2)$.

Calculate this integral numerically to high number of digits, e.g. 40 digits.

$$I \approx 0.3395454690873598695906678484608602061388$$

The PSLQ algorithm yields:

$$I = -\frac{1}{12}\ln^4(2) + \frac{\pi^4}{60} + \frac{3}{4}\ln(2)\zeta_3 + \frac{1}{12}\ln^2(2)\pi^2 - 2\mathrm{Li}_4\left(\frac{1}{2}\right)$$

$$\zeta_{2k} = (-1)^{k-1}\frac{(2\pi)^{2k}B_{2k}}{2(2k)!}; \quad B_n \quad \text{[Bernoulli number]}$$

The method can be applied to all number classes, for which the bases are known.

1. multiple zeta values
2. classes of generalized harmonic numbers
3. cyclotomic multiple zeta values
4. elliptic numbers
5. etc. etc.

In various cases one can work with successive set extensions.

The method is important in determining the value of master integrals in the zero-scale case. This e.g. applies also to the master integrals appearing in the post-Newtonian expansion. [Foffa et al. 1612.00482, 1902.10571, Blümlein et al. 1902.11180].

Similarly, constants of that kind form also initial values for differential and difference equations for single variate integrals.

It is often easier to calculate Mellin moments for a quantity for fixed values of N than to derive the relation for general values of N in the first place. If the quantity under consideration is known to be recurrent than its difference equation is of finite order and degree.

$$\exists \quad \sum_{k=0}^{O} P_k^{(l)}(N) F(k+N) = 0; \qquad \max\{l\} \text{ - degree; } O \text{ - order}$$

Example:

$$-(N+1)^3 F(N) - (3N^2 - 9N - 7) F(N+1) + (N+2)^3 F(N+2) = 0$$

$$F(1) = 1; \quad F(2) = \frac{1}{8}$$

Solution: $\qquad F(N) = \sum_{k=1}^{N} \frac{1}{k^3} = S_3(N)$

Solution of large problems

Assume you would like to calculate the massless 3-loop Wilson coefficients in deep-inelastic scattering using this method. How many moments would you need and how do they look like ?

About 5200 moments are needed. The largest ones are ratios of #13000/#13000 digits. They can be calculated within 15 min.

After 3 weeks you will find a difference equation of degree ~ 1000 and order 35, if you have a reasonable computer (100 Gbyte RAM).

After another week you have the solution as function of N.

Problem: It is sophisticated to obtain the input a priori. Solved: JB & C. Schneider, 2017.

Recent result: a 3-loop anomalous dimension computed from scratch.

[arXiv:1701.04614, 1705.01508, 1908.00357]: For contemporary non-trivial cases we have calculated 8000 moments.

Hyperlogarithms

1. Assume that a Feynman parameterization exists, which is multilinear in all parameters.
2. Assume that the integration procedure maintains this property in one order of integrations [Fubini sequence].
3. Assume, the integral has no poles in ε; or find a method to deal with it.
4. Then: the integral can be organized fully in Hyperlogarithms.
5. Hyperlogarithms are Kummer-Poincaré-Lappo-Danielevsky-Chen-Goncharov iterated integrals over an alphabet, the letters of which contain further integration variables in the multilinear sense.
6. Very many of them have coefficient zero in the final result.
7. In various cases the multi-linearity may not persist, but a solution can be found as well in extended function spaces.

$$L_{a_1,\dots a_k}(x) = \int_0^x \frac{dx_1}{x_1 - a_1} \int_0^{x_1} \frac{dx_2}{x_2 - a_2} \dots \int_0^{x_{k-1}} \frac{dx_k}{x_k - a_k}, a_l \in \mathbb{C}$$

The IBPs deliver a vast amount of differential equations forming systems, which are nested hierarchically.

Provide boundary conditions [usually using other methods]

Perform uncoupling of these systems

• In case of complete 1st order uncoupling: $\exists$ complete solution algorithm in case of any basis choice for 1 parameter systems
All solutions are iterative integrals over whatsoever alphabet: $\int_0^x dy f_a(y) H_{\vec{b}}(y)$

• Irreducible nth order systems ($n \geq 2$): present target of research even in mathematics; good prospects in case of 2nd order systems [elliptic and ${}_2F_1$ solutions]
At least one function is given by a definite integral, others iterate on.

$\Longrightarrow$ iterated integral algebras $\Longrightarrow$ bases

Risch-algorithm: Integrability of elementary functions in terms of elementary functions.

Analyse whether or not certain indefinite integrals appearing in QFT calcultions are elements of higher pre-defined function spaces.

This can be decided by solving associated differential equation systems.

Iterative integrals over certain alphabets belong to this class and the associated alphabets are even coined in this way.

$$H_{b,\vec{a}}(x) = \int_0^x dy f_b(y) H_{\vec{a}}(y), H_\emptyset = 1$$

$$f_c(x) \in \{f_1(x), ..., f_m(x)\}$$

The algorithm tries to transforms the corresponding integrals into iterative integrals. This also allows to use algebraic shuffle relations.

The integrals can usually be traded for a lower number of sums (finite or infinite).
Solve these sums for N and/or in terms of special constants.

Principle Idea:

1. Sums may be represented in vector spaces, algebras, and finally fields/rings
2. Rephrase the sums in the setting of difference fields and rings
3. Apply telescoping, creative telescoping, and other principles in this setting to compute recurrences
4. Try to solve the recurrences; possible for most sums occurring from Feynman integrals
5. In addition, use nested sums algebras to speed up calculations

Telescoping: Find a function $g(k)$ such

$$\begin{aligned} f(k) &= g(k+1) - g(k) \\ F(N) &= \sum_{k=1}^{N} f(k) = g(N+1) - g(1) \end{aligned}$$

$\Longrightarrow$ nested sums algebras $\Longrightarrow$ bases
Sigma solves large scale problems running over months and using several hundred Gb RAM.

Almkvist-Zeilberger Algorithm

- Given a multiple integral over hyperexponential terms:

$$F(n) = \int_0^1 dx_1 ... dx_j \prod_{k=1}^{l} (P(x_i, n))^{r_k, \epsilon}, \; r_k \in \mathbb{R} \text{ and } n \in \mathbb{N} \text{ a parameter.}$$

- Find a recurrence:

$$\sum_{k=0}^{m} p_k(n, \epsilon) F(n+k) = H(n, \epsilon) \text{ with some inhomogeneity } H(n, \epsilon).$$

- Correspondingly $n \rightarrow x$, a differential equation:

$$\sum_{k=0}^{m} p_k(x, \epsilon) \frac{d^k}{dx^k} F(x) = K(x, \epsilon) \text{ with some inhomogeneity } K(x, \epsilon).$$

Either the inhomogeneities can be forced to vanish, or a hierarchy of equations has to be solved using summation techniques and DEQ-solvers (which may also be summation techniques).

The AZ-Theorem states, that for recurrent quantities in a variable N one can always find a recurrence successively. It may have of course a very high order and/or degree. This applies also to multi-integrals depending on N. Conversely, one can consider the associated differential equation to $x \in \mathbb{C}$, e.g.

All our symbolic integration codes are written in Mathematica.

- Over the years they were steadily improved and extended.

- Mathematica's rich special function implementations and the strong integrator are most helpful.

- This also applies to the math world's pages and detailed on-line tabulations of other kind.

- Freeing memory in Mathematica which is no longer used would be instrumental in some cases. We operate jobs with a RAM request of up to $\sim$ 500 Gbyte and sometimes face difficulties.

- Dynamic outsourcing to fast disc, like available in FORM, would be very helpful.

- In some cases relying heavily on very fast integer arithmetics we had to use Sage because of the size and run time requests of our current problems.

[JB, A.Maier, P. Marquard, G. Schäfer, C. Schneider 1911.04411 [gr-qc]]

$$\begin{aligned} H(\mathbf{p},\mathbf{r}) &= \sqrt{m_1^2+\mathbf{p}^2}+\sqrt{m_2^2+\mathbf{p}^2}+V(\mathbf{p},\mathbf{r}), \\ V(\mathbf{p},\mathbf{r}) &= \sum_{k=1}^{\infty} V_k(\mathbf{p})\frac{G_N^k}{|\mathbf{r}|^k}, \qquad V_k(\mathbf{p}) = \sum_{l=0}^{\infty} a_k(l)x^l, \end{aligned}$$

[Consider the PM-potentials by Z. Bern et al. (2019) [isotropic coordinates].]

Reconstruction Criteria:

1. There exist recurrences for the coefficients $a_k(l)$ in l, up to a finite number of polynomial terms in x.
2. The recurrence or its associated differential equation factorizes at first order.
3. The dependence of $V_k(x)$ on $\rho = m_1/m_2$ is rational.

The equal mass case:

$$x = \frac{\mathbf{p}^2}{m^2}$$

$$\sum_{n=0}^{o} Q_n(l) f[l+n] = 0.$$

$$\begin{aligned} V_1(x) &\simeq -1 - 7x - x^2 + x^3 - x^4 + x^5 - x^6 + x^7 - x^8 + x^9 - x^{10} + x^{11} - x^{12} + x^{13} - x^{14} \\ &\quad + x^{15} - x^{16} + x^{17} - x^{18} + x^{19} - x^{20} + O(x^{21}), \\ S_1 &= \{-1, -7, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1\}. \end{aligned}$$

$$f_1[n] + f_1[n+1] = 0,$$

$$\bar{a}_1(l) = (-1)^{l+1}.$$

$$V_k(\mathbf{p}) = p_k(x) + \sum_{l=0}^{\infty} \bar{a}_k(l) x^l,$$

$$V_1 = p_1(x) - \frac{1}{1+x}, \tag{1}$$

$$p_1(x) = -8x.$$

	order	degree	# input values
1PM	1	0	8
2PM	2	5	24
3PM	3	15	54

Table 1: Characteristics of the recurrences and number of input values for the different post-Minkowskian orders in the equal mass case.

The recurrence for V_3 reads

$$Q_1\ f_3[n] + Q_2\ f_3[n+1] + Q_3\ f_3[n+2] + Q_4\ f_3[n+3] = 0,$$

$$\begin{aligned}
Q_1 &= 126903309120 + 327090111984n + 199501827192n^2 - 15839063268n^3 \\
&+125598633964n^4 + 319201064194n^5 + 244500413870n^6 + 74947793534n^7 \\
&-2304037362n^8 - 7916007828n^9 - 1912314952n^{10} - 69778816n^{11} \\
&+36357088n^{12} + 6925120n^{13} + 938880n^{14} + 84480n^{15}, \\
Q_2 &= 120213548280 + 370215834660n + 215909250030n^2 - 222044191596n^3 \\
&-29242273581n^4 + 508977450525n^5 + 530659013385n^6 + 196490815287n^7 \\
&+1030457202n^8 - 20244382200n^9 - 5290755840n^{10} - 189159456n^{11} \\
&+115021824n^{12} + 22664640n^{13} + 2985600n^{14} + 253440n^{15}, \\
Q_3 &= -101452470840 - 39208126464n + 158218785864n^2 - 262227440529n^3 \\
&-511020293886n^4 + 6335298708n^5 + 336175588032n^6 + 179465093733n^7 \\
&+10546876182n^8 - 17246382984n^9 - 4999683456n^{10} - 174901344 * n^{11} \\
&+123193344n^{12} + 24786240n^{13} + 3154560n^{14} + 253440n^{15}, \\
Q_4 &= 185299500960 + 326857962960n + 377719564176n^2 + 3846793164n^3 \\
&-343897751366n^4 - 172310662748n^5 + 55100830352n^6 + 58117627580n^7 \\
&+6784432878n^8 - 5022804012n^9 - 1624176328n^{10} - 53736544n^{11} \\
&+44718688n^{12} + 9046720n^{13} + 1107840n^{14} + 84480n^{15}.
\end{aligned}$$

$$\bar{a}_2(l) = 6(-1)^{l+1} - \frac{\left(87 + 96l + 168l^2 + 128l^3 - 16l^4\right)}{(2l-3)(2l-1)} \left(\frac{-1}{4}\right)^{l+1} \frac{(2l)!}{(l!)^2},$$

$$\bar{a}_3(l) = \frac{1}{6}\left(-114 - 94l - 15l^2 + l^3\right)(-1)^l + \frac{\left(12 + 22l + l^2 + 14l^3 + 11l^4\right)}{(l-1)l(1+2l)(3+2l)} \frac{(-1)^l 2^{3+2l}(l!)^2}{(2l)!}$$
$$+ \frac{3(5+2l)\left(-283 - 470l - 312l^2 - 40l^3 + 16l^4\right)}{(1+l)(2+l)(-1+2l)} \frac{(-1)^l 2^{-2-2l}(2l)!}{(l!)^2}.$$

$$V_3 = -\frac{3424x^2}{3} + \frac{3\left(-1+\sqrt{1+x}\right)}{x} + \frac{80}{3}x\left(-25 + 27\sqrt{1+x}\right)$$
$$+ \frac{-1 + 8(1+x) - 23(1+x)^3 - 68(1+x)^4 + 3\left(34 + 22x - 55x^2 - 40x^3\right)(1+x)^{3/2}}{(1+x)^4}$$
$$+ \frac{2\log\left(\sqrt{x} + \sqrt{1+x}\right)\left[-11 + 16x(1+x)(-1 + 4x(1+x))\right]}{\sqrt{x}(1+x)^{3/2}}.$$

The non-equal mass case:

$$z(x) = \frac{1+\rho^2}{4\rho} + \frac{\rho x}{2} + \frac{1}{2}\sqrt{1+x}\sqrt{1+\rho^2 x}, \quad x = \frac{\mathbf{p}^2}{m_1^2}$$

$$= \frac{(E_1+E_2)^2}{4m_1 m_2}, \quad \rho = \frac{m_1}{m_2}, \quad E = E_1 + E_2, \quad M_\pm = m_1 \pm m_2$$

$$x \equiv \frac{[(1-\rho)^2 - 4\rho z][(1+\rho)^2 - 4\rho z]}{16\rho^3 z} = \frac{(E^2 - M_+^2)(E^2 - M_-^2)}{4E^2 m_1^2},$$

$$\bar{z} = z - \frac{(m_1+m_2)^2}{4m_1 m_2},$$

	order	degree	# input values
1PM	2	0	8
2PM	4	12	45
3PM	9	26	120

Table 2: Characteristics of the recurrences and number of input values for the different post-Minkowskian orders in the general case.

One performs the calculation for different values of ρ chosen as a prime number and can then reconstruct the analytic dependence on ρ, i.e. lifting the single-valued problem lifting to the two-valued problem.

The Individual Steps: from IBPs to Closed Form q-Series

- Generate the master integrals, determine their hierarchy, and look whether you have only 1st order factorization or also 2nd order terms
- The latter can be trivial in case; check whether they persist in Mellin space
- If yes, analyze the 2nd order differential equation
- One usually finds a ${}_2F_1$-solution with rational argument $r(z)$, where $r(z)$ has additional singularities, i.e. the problem is of 2nd order, but has more than 3 singularities.
- Triangle group relations may be used to map the ${}_2F_1$ depending on the rational parameters a,b,c to the complete elliptic integrals or not.
- In the latter case return to the formalism on slide 21 and stop.
- If yes, one may walk along the q-series avenue.
- Different Levels of Complexity:
  - 1st order factorization in Mellin space:

$$\mathbf{M}[\mathbf{K}(1-z)](N) = \frac{2^{4N+1}}{(1+2N)^2\binom{2N}{N}^2}; \qquad \mathbf{M}[\mathbf{E}(1-z)](N) = \frac{2^{4N+2}}{(1+2N)^2(3+2N)\binom{2N}{N}^2}$$

- Criteria by Herfurtner (1991), Movasati et al. (2009) are obeyed.
  $\Longrightarrow$ 2-loop sunrise and kite diagrams, cf. Weinzierl et al. 2014-17.
  Only $\mathbf{K}(r(z))$ and $\mathbf{K}'(r(z))$ contribute as elliptic integrals.
- Also $\mathbf{E}(r(z))$ and $\mathbf{E}'(r(z))$, square roots of quadratic forms etc. contribute (present case)
- Transform now: $x \to q$.
- The kinematic variable x:

$$k^2 = \frac{-x^3}{(1+x)^3(1-3x)} = \frac{\vartheta_2^4(q)}{\vartheta_3^4(q)}$$

$$x = \frac{\vartheta_2^2(q)}{3\vartheta_2^2(q^3)}, \quad \text{i.e.} \quad x \in [1, +\infty[$$

by a cubic transformation (Legendre-Jacobi).
[see also Borwein,Borwein: AGM; and Broadhurst (2008).]

$$x = \frac{1}{3}\frac{\eta^2(2\tau)\eta^2(3\tau)}{\eta^2(\tau)\eta^4(6\tau)}, \quad \text{singular}, \propto \frac{1}{q}$$

The Individual Steps: from IBPs to Closed Form q-Series

- Map to a Modular Form, which can be represented by Lambert Series
  - How to find the η-ratio ? $\Longrightarrow$ Many are listed as sequences in Sloan's OEIS.
  - To find a modular form, situated in a corresponding finite-dimensional vector space M_k one has to meet a series of conditions and usually split off a factor $1/\eta^k(\tau), k > 0$.
  - The remainder modular form is now a polynomial over $\mathbb{Q}$ of Lambert-Eisenstein series

$$\sum_{n=0}^{\infty} \frac{m^n q^{an+b}}{1 - q^{an+b}} \,.$$

Example:

$$\mathbf{K}(z(x)) = \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{q^k}{1 + q^{2k}}$$

  - In this case, two q series are equal, if both are modular forms, and agree in a series of k first terms, where k is predicted for each congruence sub-group of $\Gamma(N)$.

The Individual Steps: from IBPs to Closed Form q-Series

- Map Lambert-Eisenstein Series into the frame of Elliptic Polylogarithms
- Examples:

$$\begin{aligned}
\mathbf{K}(z) &= \frac{\pi}{2}\sum_{k=1}^{\infty}\frac{q^k}{1+q^{2k}} = \frac{\pi}{i}\sum_{k=1}^{\infty}\left[\mathrm{Li}_0\left(iq^k\right) - \mathrm{Li}_0\left(-iq^k\right)\right] \\
&= \frac{\pi}{4}\overline{E}_{0,0}(i,1,q), \\
q\frac{\vartheta_4'(q)}{\vartheta_4(q)} &= -\frac{1}{2}\left[\mathrm{ELi}_{-1;0}(1;1;q) + \mathrm{ELi}_{-1;0}(-1;1;q)\right] \\
&+ \left[\mathrm{ELi}_{0;0}(1;q^{-1};q) + \mathrm{ELi}_{0;0}(-1;q^{-1};q)\right] \\
&- \left[\mathrm{ELi}_{-1;0}(1;q^{-1};q) + \mathrm{ELi}_{-1;0}(-1;q^{-1};q)\right].
\end{aligned}$$

- New type of elliptic polylogarithm, e.g.: $\mathrm{ELi}_{-1;0}(-1;q^{-1};q)$, $\quad y = y(q)$!
- Argument synchronization necessary: $-q \to q$, $\quad q^k \to q$ (cyclotomic).

- Terms to be translated:
  - rational functions in x
  - $\mathbf{K}, \mathbf{E}$
  - $\sqrt{(1-3x)(1+x)}$
  - $H_{\vec{a}}(x)$

Examples:

$$\mathbf{E}(k^2) = \mathbf{K}(k^2) + \frac{\pi^2 q}{\mathbf{K}(k^2)} \frac{d}{dq} \ln\left[\vartheta_4(q)\right]$$

$$\mathbf{E}'(k^2) = \frac{\pi}{2\mathbf{K}(k^2)} \left[1 + 2\ln(q)\; q\frac{d}{dq} \ln\left[\vartheta_4(q)\right]\right].$$

$$\frac{1}{\mathbf{K}(k^2)} = \frac{2}{\pi\eta^{12}(\tau)} \left\{\frac{5}{48}\left\{1 - 24\mathrm{ELi}_{0;-1}(1;1;q) - 4\left[1 - \frac{3}{2}\left[\mathrm{ELi}_{0;-1}(1;1;q) + \mathrm{ELi}_{0;-1}(1;i;q)\right.\right.\right.\right.$$
$$\left.\left.\left.\left. + \mathrm{ELi}_{0;-1}(1;-1;q) + \mathrm{ELi}_{0;-1}(1;-i;q)\right]\right]\right\}\left\{-1 + 4\left[-\frac{1}{2}\left[\mathrm{ELi}_{-2;0}(i;1/q;q)\right.\right.\right.\right.$$
$$\left. + \mathrm{ELi}_{-2,0}(-i;1/q;q)\right] + \left[\mathrm{ELi}_{-1;0}(i;1/q;q) + \mathrm{ELi}_{-1;0}(-i;1/q;q)\right] - \frac{1}{2}\left[\mathrm{ELi}_{0,0}(i;1/q;q)\right.$$

$$+\mathrm{ELi}_{0,0}(-i;1/q;q)\Big]\Big]\Bigg\} - \frac{1}{16}\Bigg\{5 + 4\Big[-\frac{1}{2}\Big[\mathrm{ELi}_{-4;0}(i;1/q;q) + \mathrm{ELi}_{-4;0}(-i;1/q;q)\Big]$$
$$+2\Big[\mathrm{ELi}_{-3;0}(i;1/q;q) + \mathrm{ELi}_{-3,0}(-i;1/q;q)\Big] - 3\Big[\mathrm{ELi}_{-2;0}(i;1/q;q)$$
$$+\mathrm{ELi}_{-2,0}(-i;1/q;q)\Big] + 2\Big[\mathrm{ELi}_{-1;0}(i;1/q;q) + \mathrm{ELi}_{-1;0}(-i;1/q;q)\Big]$$
$$-\frac{1}{2}\Big[\mathrm{ELi}_{0;0}(i;1/q;q) + \mathrm{ELi}_{0,0}(-i;1/q;q)\Big]\Big\}\Bigg\}$$

$$\begin{aligned} H_{-1}(x) = \ln(1+x) &= -\ln(3q) - \overline{E}_{0;-1;2}(-1;-1;q) + \overline{E}_{0;-1;2}(\rho_6;-1;q) \\ &\quad -\overline{E}_{0;-1;2}(\rho_3;-i;q) - \overline{E}_{0;-1;2}(\rho_3;i;q) \\ H_1(x) &= -\left. H_{-1}(x)\right|_{q\to -q} + 2\pi i, etc.; \quad \rho_m = exp(2\pi i/m) \end{aligned}$$

$$I(q) = \frac{1}{\eta^k(\tau)} \cdot \mathbf{P}\left[\ln(q), \mathrm{Li}_0(q^m), \mathrm{ELi}_{k,l}(x,y,q), \mathrm{ELi}_{k',l'}(x,q^{-1},q)\right]$$

$$\int \frac{dq}{q} I(q)$$

is usually not an elliptic polylogarithm, due to the η-factor, but a higher transcendental function in q.

We are still in the unphysical region and have to map back to $x \in [0,1]$. .

$$\Delta\rho = \frac{3G_F m_t^2}{8\pi^2\sqrt{2}}\left(\delta^{(0)}(x) + \frac{\alpha_s}{\pi}\delta^{(1)}(x) + \left(\frac{\alpha_s}{\pi}\right)^2 \delta^{(2)}(x) + \mathcal{O}(\alpha_s^3)\right)$$

$$\begin{aligned}
\delta^{(2)}(x) \quad = \quad & \cdots + C_F\left(C_F - \frac{C_A}{2}\right)\left[\frac{11-x^2}{12(1-x^2)^2} f_{8a}(x) + \frac{9-x^2}{3(1-x^2)^2} f_{9a}(x) + \frac{1}{12} f_{10a}(x)\right. \\
& \left. + \frac{5-39x^2}{36(1-x^2)^2} f_{8b}(x) + \frac{1-9x^2}{9(1-x^2)^2} f_{9b}(x) + \frac{x^2}{12} f_{10b}(x)\right] \\
& + \frac{C_F T_F}{9(1-x^2)^3}\left[(5x^4 - 28x^2 - 9) f_{8a}(x) + \frac{1-3x^2}{3x^2}(9x^4 + 9x^2 - 2) f_{8b}(x)\right. \\
& \left. + (9-x^2)(x^4 - 6x^2 - 3) f_{9a}(x) + \frac{1-9x^2}{3x^2}(3x^4 + 6x^2 - 1) f_{9b}(x)\right]
\end{aligned}$$

Insert the respective functions, which are the 2nd order solutions and perform deep enough Taylor expansions in the respective regions in x.

What can be achieved by all that?

Quantum Field Theory:

- A lot of integration technology has been created for many analytic precision calculations for the Large Hadron Collider and the planned International Linear Collider.
- The results allow also for many advanced solutions in combinatorics and number theory.
- Within elementary particle physics the present results allow to improve the precision of two fundamental constants of the Standard Model:

$$\frac{\delta\alpha_s(M_Z)}{\alpha_s(M_Z)} < 1\% \qquad\qquad \delta m_c < 20\text{MeV}[1.6\%]$$

which may have consequences for various proposed extensions of the SM.

Gravity:

Many of the above techniques apply also in the effective field theory approach to gravity and proof to be very efficient there, in particular one wants to address even higher orders.

Conclusions

1. Current problems at the precision frontier of Quantum Field Theory request advanced integration (anti-differentiation) methods.
2. There are standardized chains down to the master integrals, with excellent computer algebraic implementations.
3. The calculation of the remainder integrals forms the current challenge.
4. The first order factorizable cases in case of nested sums or difference equations are completely understood and there are good implementations to solve these cases.
5. There are techniques for 2nd order factorization cases, still in course of development.
6. More involved structures require a lot of further efforts.
7. Massive problems (from 1-scale also in gravity) map to more involved function spaces.
8. The present techniques form one important ingredient to proceed to higher post-Newtonian and post-Minkowskian orders using effective field theory methods.

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