Harmonic, Generalized Harmonic, Cyclotomic and Binomial Sums, Polylogarithms and Special Numbers

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Contents

Introduction

- Harmonic sums and harmonic polylogarithms have made many loop calculations simpler since the late 1990ies.
- $\Longrightarrow$ adequate structures for Feynman diagrams
- iterated parameter integrals of the Volterra type
- naturally emerging in the ε-expansion
- massless and single mass cases 2 Loops: only usual harmonic sums/polylogarithms
- 3 Loops: first traces of generalized harmonic sums
- massive calculations: 4th, 6th root of unity weights, ...
- Generalization of harmonic sums and polylogarithms required

$\Longrightarrow$ Instrumental for the contemporary massless and massive 3-Loop Calculations.

Built in: `HarmonicSums`, Thesis of J. Ablinger (2012)
The code `Sigma` has been extensively used and generalized.

Introduction

Harmonic sums: (Vermaseren; JB, Kurth, 1998)

$$S_{b,\vec{a}}(N) = \sum_{k=1}^{N} \frac{(\mathrm{sign}(b))^k}{k^{|b|}} S_{\vec{a}}(k)\ , \quad S_{\emptyset}(N) = 1\ , \quad b, a_i \in \mathbb{Z}\backslash\{0\}.$$

Generalized harmonic sums: (Moch, Uwer, Weinzierl, 2001; Ablinger, JB, Schneider, 2013)

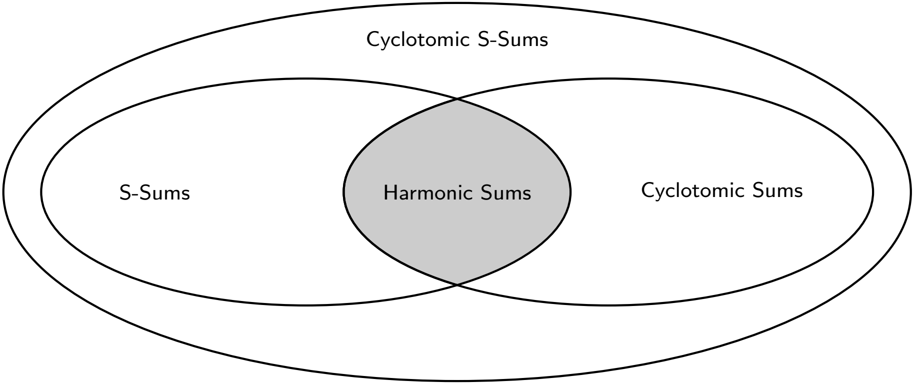
$$S_{b,\vec{a}}(\zeta, \vec{\xi}; N) = \sum_{k=1}^{N} \frac{\zeta^k}{k^b} S_{\vec{a}}(\vec{\xi}; k)\ , \quad b, a_i \in \mathbb{N}_+; \zeta, \xi_i \in \mathbb{R}^* = \mathbb{R}\backslash\{0\}$$

Known examples are related to the second index set $\xi_i \in \{1, -1, 1/2, -1/2, 2, -2\}$.
Other Summands occurring:

$$\frac{(\pm 1)^k}{(l \cdot k + m)^n}\ , \quad \text{with} \quad l, m, n \in \mathbb{N}_+.$$

New Mellin variable:

$$N \to k \cdot N\ ,$$


Cyclotomic S-Sums
S-Sums
Harmonic Sums
Cyclotomic Sums

Introduction

Connection to iterated Integrals:

$$S_{\vec{a}}(\vec{b}; N) = \sum_c \int_0^1 dx\ x^N\ H_{\vec{c}}(x)$$

$$\text{Mellin Transform:}\quad H_{\vec{c}}(x) = \int_0^x \frac{dx_1}{x_1 - c_1} \dots \int_0^{x_{k-1}} dx_k \frac{dx_1}{x_1 - c_k}$$

Special Numbers:

$$S_{\vec{a}}(\vec{b}; \infty) \qquad \text{or} \qquad H_{\vec{c}}(1)$$

Algebraic Relations: (JB, 2003)

All these objects form stuffle (Sums) and shuffle (Iter. Integrals) Algebras. The special numbers obey both algebras.
MZV's and Euler sums: (JB, Broadhurst, Vermaseren, 2009).
These relations also hold for all extensions discussed below.

$$H_a(x) \cdot H_{bcd}(x) = H_a(x) ⧢ H_{bcd}(x) = H_{abcd}(x) + H_{bacd}(x) + H_{bcad}(x) + H_{bcda}(x) \quad \text{etc.}$$

Structural Relations: (JB, 2009)

Specific relations between these quantities

- Sums: differentiation for N; multiple argument relations (Harmonic Sums, Generalized Harmonic Sums, Cyclotomic Sums)
- HPLs: conformal argument mapping

$$x \to \frac{1-x}{1+x};$$

  and other special argument relations
- Euler-Zagier values: doubling relation (w=8); generalized doubling relations (w=12) (JB, Broadhurst, Vermaseren, 2009)

Cyclotomic Sums: Basic Formalism

$$
\begin{aligned}
S_{\{a_1,b_1,c_1\},\ldots,\{a_l,b_l,c_l\}}(s_1,\ldots,s_l;N) &= \sum_{k_1=1}^{N} \frac{s_1^k}{(a_1 k_1 + b_1)^{c_1}} S_{\{a_2,b_2,c_2\};\ldots;\{a_l,b_l,c_l\}}(s_2,\ldots,s_l;k_1), \\
S_\emptyset &= 1, \quad a_i, c_i \in \mathbb{N}_+, b_i \in \mathbb{N}, \quad s_i = \pm 1, a_i > b_i
\end{aligned}
$$

Summand representation:

$$
\begin{aligned}
\frac{(\pm 1)^k}{ak+b} &= \int_0^1 dx\ x^{ak+b-1} (\pm 1)^k \\
\frac{(\pm 1)^k}{(ak+b)^c} &= \int_0^1 \frac{dx_1}{x_1} \int_0^{x_1} \frac{dx_2}{x_2} \ldots \int_0^{x_{c-2}} \frac{dx_{c-1}}{x_{c-1}} \int_0^{x_{c-1}} dx_c\ x_c^{ak+b-1} (\pm 1)^k,
\end{aligned}
$$

$$
\sum_{k=1}^{l} (\pm 1)^k x^{ak+b-1} = x^{a+b-1} \frac{(\pm x^a)^{l+1} - 1}{(\pm x^a) - 1}\ .
$$

Let us illustrate the principle steps in case of the following example :

$$S_{\{3,2,2\},\{2,1,1\}}(1,-1;N) = \sum_{k=1}^{N} \frac{1}{(3k+2)^2} \sum_{l=1}^{k} \frac{(-1)^l}{(2l+1)} .$$

$$S_{\{3,2,2\},\{2,1,1\}}(1,-1;N) = \sum_{k=1}^{N} \int_0^1 dx \frac{x^2}{x^2+1} \frac{(-x^2)^k+1}{(3k+2)^2} .$$

Setting $x = y^3$ one obtains

$$\begin{aligned}
S_{\{3,2,2\},\{2,1,1\}}(1,-1;N) &= 12 \int_0^1 dy \frac{y^8}{y^6+1} \sum_{k=1}^{N} \frac{(-y^6)^k-1}{(6k+4)^2} \\
&= 12 \int_0^1 dy \frac{y^4}{y^6+1} \Bigg\{ \int_0^y \frac{dz}{z} \int_0^z dt\, t^9 \frac{(-t^6)^N-1}{t^6+1} \\
&\qquad -y^4 \int_0^1 \frac{dz}{z} \int_0^z dt\, t^9 \frac{t^{6N}-1}{t^6-1} \Bigg\} \\
&= 12 \int_0^1 dy \frac{y^4}{y^6+1} \int_0^y \frac{dz}{z} \int_0^z dt\, t^9 \frac{(-t^6)^N-1}{t^6+1} \\
&\qquad -(4-\pi) \int_0^1 \frac{dz}{z} \int_0^z dt\, t^9 \frac{t^{6N}-1}{t^6-1} \Bigg\} .
\end{aligned}$$

In general, the polynomials

$$x^a - 1$$

decompose in a product of cyclotomic polynomials, except for $a = 1$ for which the expression is $\Phi_1(x)$. Moreover, the polynomials

$$x^a + 1 = \frac{x^{2a} - 1}{x^a - 1}$$

are either cyclotomic for $a = 2^n, n \in \mathbb{N}$ or decompose into products of cyclotomic polynomials in other cases. All factors divide $(x^a)^l - 1$, resp. $(-x^a)^l - 1$.

$$\begin{aligned}
\Phi_1(x) &= x - 1 \\
\Phi_2(x) &= x + 1 \\
\Phi_3(x) &= x^2 + x + 1 \\
\Phi_4(x) &= x^2 + 1 \\
\Phi_5(x) &= x^4 + x^3 + x^2 + x + 1 \\
\Phi_6(x) &= x^2 - x + 1 \\
\Phi_7(x) &= x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\
\Phi_8(x) &= x^4 + 1 \\
\Phi_9(x) &= x^6 + x^3 + 1 \\
\Phi_{10}(x) &= x^4 - x^3 + x^2 - x + 1 \\
\Phi_{11}(x) &= x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\
\Phi_{12}(x) &= x^4 - x^2 + 1, \quad \text{etc.}
\end{aligned}$$

Cyclotomic Harmonic Polylogarithms

To account for the newly emerging sums in perturbative calculations in Quantum Field Theory we introduce Poincaré-iterated integrals over the alphabet $\mathfrak{A}$

$$\mathfrak{A} = \left\{ \frac{1}{x} \right\} \cup \left\{ \frac{x^l}{\Phi_k(x)} \middle| k \in \mathbb{N}_+, 0 \leq l < \varphi(k) \right\},$$

where $\Phi_k(x)$ denotes the kth cyclotomic polynomial and $\varphi(k)$ denotes Euler's totient function.

$$\Phi_n(x) = \frac{x^n - 1}{\prod\limits_{d|n, d<n} \Phi_d(x)}, \qquad d, n \in \mathbb{N}_+,$$

The alphabet $\mathfrak{A}$ is an extension of the alphabet

$$\mathfrak{A}_{\mathrm{H}} = \left\{ \frac{1}{x}, \frac{1}{\Phi_1(x)}, \frac{1}{\Phi_2(x)} \right\},$$

generating the usual harmonic polylogarithms (Remiddi, Vermaseren, 1999)

A shorthand notation for the letters of $\mathfrak{A}$:

$$\begin{aligned} f_0^0(x) &= \frac{1}{x} \\ f_k^l(x) &= \frac{x^l}{\Phi_k(x)}, \qquad k \in \mathbb{N}_+, l \in \mathbb{N}, l < \varphi(k)\ . \end{aligned}$$

Examples: (N. Nielsen, 1906)

$$\begin{aligned} \frac{1}{2}\beta\left(\frac{x}{2}\right) &= \int_0^1 dt \frac{t^{x-1}}{t^2+1} \\ \beta\left(\frac{x}{3}\right) &= \beta(x) - \int_0^1 dt\ t^{x-1}\frac{t-2}{t^2-t+1}\ , \\ \beta(x) &= \frac{1}{2}\left[\Psi\left(\frac{x+1}{2}\right) - \Psi\left(\frac{x}{2}\right)\right]\ . \end{aligned}$$

We form the Poincaré iterated integrals

$$
\begin{aligned}
C^{l_1,\ldots,l_m}_{k_1,\ldots,k_m}(z) &= \frac{1}{m!}\ln^m(x) && \text{if } (l_1,\ldots,l_m) = (0,\ldots,0),\\
& && (k_1,\ldots,k_m) = (0,\ldots,0),\\
C^{l_m}_{k_m}(z) &= \int_0^z dx\ f^{l_m}_{k_m}(x) && \text{if } k_m \neq 0,\\
C^{l_1,\ldots,l_m}_{k_1,\ldots,k_m}(z) &= \int_0^z dx\ f^{l_1}_{k_1}(x)\ C^{l_2,\ldots,l_m}_{k_2,\ldots,k_m}(x) && \text{if } (k_1,\ldots,k_m) \neq (0,\ldots,0),
\end{aligned}
$$

and $C^{\vec{l}}_{\vec{a}}(z)$ denotes cyclotomic harmonic polylogarithms of weight $\mathrm{w} = m$. They form a shuffle algebra by multiplication

$$
C^{\vec{a_2}}_{\vec{a_1}}(z) \cdot C^{\vec{b_2}}_{\vec{b_1}}(z) = C^{\vec{a_2}}_{\vec{a_1}}(z) ⧢ C^{\vec{b_2}}_{\vec{b_1}}(z) = \sum_{\left[\begin{array}{c}\vec{c}_2\\ \vec{c}_1\end{array}\right] \in \left[\begin{array}{c}\vec{a}_2\\ \vec{a}_1\end{array}\right] ⧢ \left[\begin{array}{c}\vec{b}_2\\ \vec{b}_1\end{array}\right]} C^{\vec{c_2}}_{\vec{c_1}}(z)
$$

of M^{w} elements at weight w, where M denotes the number of chosen letters from $\mathfrak{A}$.

$$
N^{\mathrm{basic}}(\mathrm{w}) = \frac{1}{\mathrm{w}} \sum_{d|\mathrm{w}} \mu\left(\frac{\mathrm{w}}{d}\right) M^d, \mathrm{w} \geq 1
$$

weight	letters						
	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8
2	1	3	6	10	15	21	28
3	2	8	20	40	70	112	168
4	3	18	60	150	315	588	1008
5	6	48	204	624	1554	3360	6552
6	9	116	670	2580	7735	19544	43596
7	18	312	2340	11160	39990	117648	299592
8	30	810	8160	48750	209790	729300	2096640

Table: Number of basic cyclotomic harmonic polylogarithms in dependence of the number of letters and weight.

$f_{i_a}^{j_b}$ representation:

$$\frac{y^4}{y^6+1} = \frac{1}{3}\left[f_4^0(y) - f_{12}^0(y) + 2f_{12}^2(y)\right] .$$

With integration by parts one obtains the following Mellin transforms of argument $6N$ of cyclotomic harmonic polylogarithms $C_{k_1,\ldots,k_m}^{l_1,\ldots,l_m}(x)$ weighted by the letters $f_l^k(x)$ of the alphabet $\mathfrak{A}$:

$$
\begin{aligned}
&S_{\{3,2,2\},\{2,1,1\}}(1,-1;N) = \\
&\frac{1}{6}(4-\pi)\int_0^1 dx x^3 (x^{6N}-1)\left[6 + f_1^0(x) - f_2^0(x) - 2f_3^0(x)\right. \\
&\qquad\qquad \left. -f_3^1(x) - 2f_6^0(x) + f_6^1(x)\right] C_0^0(x) \\
&-2\int_0^1 dx x^3 \left[(-1)^N x^{6N} - 1\right]\left[3 - f_4^0(x) - 2f_{12}^0(x) + 2f_{12}^2(x)\right] C_0^0(x) \\
&-\frac{4}{3}\left[C_{0,4}^{0,0}(1) - C_{0,12}^{0,0}(1) + 2C_{0,12}^{0,2}(1)\right]\int_0^1 dx x^3 \left[(-1)^N x^{6N} - 1\right] \\
&\qquad\qquad \times \left[3 - f_4^0(x) - 2f_{12}^0(x) + 2f_{12}^2(x)\right] \\
&+\frac{4}{3}\int_0^1 dx x^3 \left[(-1)^N x^{6N} - 1\right]\left[C_{0,4}^{0,0}(x) - C_{0,12}^{0,0}(x) + 2C_{0,12}^{0,2}(x)\right] \\
&\qquad\qquad \times \left[3 - f_4^0(x) - 2f_{12}^0(x) + 2f_{12}^2(x)\right] .
\end{aligned}
$$

$$C_{0,4}^{0,0} = -\mathbf{C}, \qquad C_{0,12}^{0,0(2)} = \alpha_1 \psi'(1/12) + \alpha_2 \psi'(5/12)$$

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Cyclotomic Harmonic Sums

(i) The Single Sums:

$$S_{l,m,n}(N) = \sum_{k=0}^{N} \frac{(\pm 1)^k}{(l \cdot k + m)^n}; \qquad \phi_k(l,N) = \int_0^1 dx x^N f_k^l(x)$$

The summation package Sigma reduces to :

$$S_{-1}(N), S_{-1}(3N), S_{-1}(5N), S_1(N), S_1(2N), S_1(3N), S_1(4N), S_1(5N), S_1(6N),$$
$$\phi_3(3N)_+, \phi_3(6N)_+, \phi_4(2N), \phi_4(4N), \phi_4(6N), \phi_5(5N)_+, \phi_5(1,5N)_+,$$
$$\phi_5(2,5N)_+, \phi_6(3N), \phi_8(4N), \phi_8(2,4N), \phi_{10}(5N), \phi_{10}(1,5N), \phi_{10}(2,5N), \phi_{12}(6N).$$

and the sums at $N \to \infty$

$$\sigma_{\{1,0,-1\}}, \sigma_{\{2,1,-1\}}, \sigma_{\{3,1,-1\}}, \sigma_{\{4,1,-1\}}, \sigma_{\{4,3,-1\}}, \sigma_{\{5,1,-1\}}, \sigma_{\{5,2,-1\}}, \sigma_{\{5,3,-1\}} \sigma_{\{6,1,-1\}}$$

Asymptotic representations $|N| \to \infty$ are derived easily.
Needed within the summation calculus.

(ii) Relations between sums:

- Differentiation w.r.t. N, (D)
- Stuffle Algebra (A)
- Synchronization (M)
- Duplication Relations: (H_1, H_2)

(iii) Nested Sums: Consider:

$$\frac{1}{k^{l_1}}, \qquad \frac{(-1)^k}{k^{l_2}}, \qquad \frac{1}{(2k+1)^{l_3}}, \qquad \frac{(-1)^k}{(2k+1)^{l_4}} \qquad (*)$$

w	N_S	H_1	H_1, H_2	H_1, M	H_1, H_2, M	D	H_1, H_2, M, D	A	H_1, H_2, M, A	A, D	all
1	4	3	3	2	2	4	2	4	2	4	2
2	20	18	17	16	15	16	13	10	8	6	6
3	100	96	93	92	89	80	74	40	35	30	27
4	500	492	485	484	477	400	388	150	142	110	107
5	2500	2484	2469	2468	2453	2000	1976	624	607	474	465

: Reduction of the number of cyclotomic harmonic sums N_S over the elements at given weight w by applying the three multiple argument relations (H_1, H_2, M), differentiation w.r.t. to the external sum index N, (D), and the algebraic relations (A). A sequence of symbols corresponds to the combination of these relations.

Explicit counting relations for the number of basis elements given in the above Table can be derived :

$$
\begin{aligned}
N_A(w) &= \frac{1}{w}\sum_{d|w}\mu\left(\frac{w}{d}\right)5^d \\
N_D(w) &= N_S(w) - N_S(w-1) = 16\cdot 5^{w-2} \\
N_{H_1}(w) &= N_S(w) - 2^{w-1} = 4\cdot 5^{w-1} - 2^{w-1} \\
N_{H_1H_2}(w) &= N_S(w) - (2\cdot 2^{w-1}-1) = 4\cdot 5^{w-1} - (2\cdot 2^{w-1}-1) \\
N_{H_1M}(w) &= N_S(w) - 2\cdot 2^{w-1} = 4\cdot 5^{w-1} - 2\cdot 2^{w-1} \\
N_{H_1H_2M}(w) &= N_S(w) - (3\cdot 2^{w-1}-1) = 4\cdot 5^{w-1} - (3\cdot 2^{w-1}-1) \\
N_{AD}(w) &= N_A(w) - N_A(w-1) \\
N_{AH_1H_2M}(w) &= \frac{1}{w}\sum_{d|w}\mu\left(\frac{w}{d}\right)5^d - \left(3\cdot\frac{1}{w}\sum_{d|w}\mu\left(\frac{w}{d}\right)2^d - 1\right) \\
N_{DH_1H_2M}(w) &= N_{H_1H_2M}(w) - N_{H_1H_2M}(w-1) = 16\cdot 5^{w-2} - 3\cdot 2^{w-2} \\
N_{ADH_1H_2M}(w) &= N_{AH_1H_2M}(w) - N_{AH_1H_2M}(w-1)
\end{aligned}
$$

Here μ denotes the Möbius function.

Special Values: Single Sums

$w = 1$

$$\psi\left(\frac{p}{q}\right) = -\gamma_E - \ln(2q) - \frac{\pi}{2}\cot\left(\frac{p\pi}{q}\right) + 2\sum_{k=1}^{[(q-1)/2]}\cos\left(\frac{2\pi kp}{q}\right)\ln\left[\sin\left(\frac{\pi k}{q}\right)\right]$$

$$\psi\left(\frac{1}{n}\right) = -n\left(\gamma_E + \ln(n)\right) - \sum_{k=2}^{n}\psi\left(\frac{k}{n}\right).$$

Further simplification if the regular q–polygon is constructible

$$q \in \{2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, 30, 32, 34, 40, 48, 51, 60, 64, \ldots\}$$

Consider the above alphabet $(*)$.
The basis elements are:

$$\sigma_0, \quad \ln(2), \quad \pi$$

$w \geq 2, l \leq 6$:

$$\zeta_{2k+1}, \psi^{(2k+1)}\left(\frac{1}{3}\right), \mathrm{Ti}_{2k}(1), \psi^{(k)}\left(\frac{1}{5}\right), \psi^{(2k+1)}\left(\frac{2}{5}\right), \psi^{(k)}\left(\frac{1}{8}\right), \psi^{(2k)}\left(\frac{1}{12}\right)$$

$$
\begin{aligned}
\mathrm{Ti}_l(1) &= \beta_D(l) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^l} \\
\mathrm{Ti}_2(1) &= \mathbf{C}
\end{aligned}
$$

Some new numbers occurring at $w = 1$:

$$
\ln(3), \ln(\sqrt{2}-1), \ln(\sqrt{3}-1), \ln(\sqrt{5}-1), \ldots
$$

$+$ algebraic (irrational) numbers

$w > 1$, higher depth, alphabet $(*)$

weight	N_S	A	SH	A + SH	A + SH + H_1	A + SH + H_1 + H_2	A + SH + H_1 + H_2 + M
1	4	4	4	4	4	3	3
2	20	10	13	3	3	2	1
3	100	40	46	6	6	5	3
4	500	150	163	10	10	9	6
5	2500	624	650	21	21	19	13
6	12500	2580	2635	36	36	34	25

Some basis elements:

$$\begin{aligned}
\sigma_{\{2,1,-2\}} &= -1 + \mathbf{C} \\
\sigma_{\{1,0,-2\},\{2,1,-1\}} &= \frac{\pi^2}{12} - \frac{\pi^3}{48} + \frac{1}{2}\int_0^1 dx \frac{\sqrt{x}}{x+1}\mathrm{Li}_2(x) \\
\sigma_{\{2,1,-2\},\{1,0,-1\}} &= -\mathbf{C}\ln(2) + \int_0^1 dx \frac{1}{1+x}\frac{\chi_2(\sqrt{x})}{\sqrt{x}} \\
\chi_\nu(x) &= \frac{1}{2}\left[\mathrm{Li}_\nu(x) - \mathrm{Li}_\nu(-x)\right]
\end{aligned}$$

Infinite Sums with more Cyclotomic Letters

$w = 1$

$$\frac{(\pm 1)^k}{(lk+m)^n}, \qquad 1 \le n \le 2, 1 \le l \le 20, m < l \ .$$

l	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
sums	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40
basis	2	3	4	5	6	6	8	9	8	10	12	10	14	14	11	17	18	14	20	18
new basis sums	2	1	2	2	4	1	6	4	4	3	10	2	12	5	3	8	16	4	18	6

The number of the w = 1 cyclotomic harmonic sums (1) up to l = 20, the basis elements at fixed value of l, and the new basis elements in ascending sequence.

$$\sigma_{\{1,0,1\}}, \sigma_{\{1,0,-1\}}, \sigma_{\{2,1,-1\}}, \sigma_{\{3,1,1\}}, \sigma_{\{3,1,-1\}}, \sigma_{\{4,1,-1\}}, \sigma_{\{4,3,-1\}},$$
$$\sigma_{\{5,1,1\}}, \sigma_{\{5,1,-1\}}, \sigma_{\{5,2,-1\}}, \sigma_{\{5,3,-1\}}, \sigma_{\{6,1,-1\}} \ , \cdots$$

$w = 2$

l	N_S	SH	A	A + SH	A + SH + H_1	A + SH + H_1 + H_2	A + SH + H_1 + H_2 + M
1	6	4	3	1	1	1	1
2	20	13	10	3	3	2	1
3	42	27	21	7	6	6	5
4	72	46	36	12	11	10	3
5	110	70	55	19	17	17	16
6	156	99	78	27	25	24	5
7	210	133	105	37	34	34	33
8	272	172	136	48	45	44	12
9	342	216	171	61	57	57	52
10	420	265	210	75	71	70	22
11	506	319	253	91	86	86	85
12	600	378	300	108	103	102	21
13	702	442	351	127	121	121	120
14	812	551	406	147	141	140	49
15	930	585	465	169	162	162	145
16	1056	664	528	192	185	184	50
17	1190	748	595	217	209	209	208
18	1332	837	666	243	235	234	63
19	1482	931	741	271	262	262	261
20	1640	1030	820	300	291	290	74

Number of basis elements of the $w = 2$ cyclotomic harmonic sums up to cyclotomy $l = 20$ after applying the quasi-shuffle algebra of the sums (A), the shuffle algebra of the cyclotomic harmonic polylogarithms (SH), and the three multiple argument relations (H_1, H_2, M) of the sums.

Counting Relations:

$$
\begin{aligned}
\mathrm{N_S}(l) &= 2l(2l+1) \\
\mathrm{N_A}(l) &= l(2l+1) \\
\mathrm{N_{SH}}(l) &= \frac{(5l+3)l}{2} \\
\mathrm{N_{A,SH}}(l) &= \frac{6l^2+1-(-1)^l}{8} \\
\mathrm{N_{A,SH,H_1}}(l) &= \frac{6l^2-4l+7-(-1)^l}{8} \\
\mathrm{N_{A,SH,H_1,H_2}}(l) &= \frac{6l^2-4l+3(1-(-1)^l)}{8} \\
\mathrm{N_{A,SH,H_1,H_2}}(l) &= \frac{3}{4}l^2-12l+if(modp(1,2)=0,1,3/4) \quad (Broadhurst)
\end{aligned}
$$

Generalized Harmonic Sums at Roots of Unity

$$\lim_{N\to\infty} S_{k_1,\dots,k_m}(x_1,\dots,x_m;N) = \sigma_{k_1,\dots,k_m}(x_1,\dots,x_m), x_j \in \mathcal{C}_n, n \geq 1, k_1 \neq 1 \text{ for } x_1 = 1$$

$$\mathcal{C}_n \in \{e_n | e_n^n = 1, e_n \in \mathbb{C}\}$$

$$\begin{aligned}
\sigma_{\rm w}(x) &= \mathrm{Li}_{\rm w}(x), \qquad \mathrm{w} \in \mathbb{N}, \mathrm{w} \geq 1 \\
\sigma_1(x) = \mathrm{Li}_1(x) &= -\ln(1-x) \\
\sigma_{1,1}(x,y) &= \mathrm{Li}_2(x) + \frac{1}{2}\ln^2(1-x) + \mathrm{Li}_2\left(-\frac{x(1-y)}{1-x}\right) \\
\sigma_{1,1}(x,x^*) &= \mathrm{Li}_2(x) + \frac{1}{2}\ln^2(1-x) + \zeta_2 \\
\mathrm{Li}_{\rm w}(x) &= \mathrm{Li}_{\rm w}^*(x^*) \\
\sigma_{1,1}(x,y) + \sigma_{1,1}(y,x) &= \ln(1-x)\ln(1-y) + \mathrm{Li}_2(xy)\ .
\end{aligned}$$

Root structure $l = 12$:

$$\left\{ e_{12}^k \Big|_{k=1}^{12} \right\} \equiv \left\{ e_{12}, e_6, e_4, e_3, e_{12}^5, e_2, e_{12}^{5\,*}, e_3^*, e_4^*, e_6^*, e_{12}^*, e_1 \right\}\ .$$

$$\mathrm{Im}\left[\mathrm{Li}_n(e_l^k)\right] = r_{n,l,k}\pi^n \text{ for } n \text{ odd}, \ \mathrm{Re}\left[\mathrm{Li}_n(e_l^k)\right] = r_{n,l,k}\pi^n \text{ for } n \text{ even}, r_{n,l,k} \in \mathbb{Q}$$

$w = 1$

l	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
basis	0	1	2	2	3	3	4	3	4	4	6	4	7	5	6	5	9	5	10	6
Racinet '01	0	1	2	2	3	3	4	3	5	4	6									
new elements	0	1	2	0	2	0	3	1	2	0	5	1	6	0	2	2	8	0	9	2

The number of the basis elements spanning the w=1 cyclotomic harmonic polylogarithms at lth roots of unity up to 20.

Basis Elements:

$$\ln(2); \ln(3), \pi; \mathrm{Re}(\mathrm{Li}_1(e_5)), \mathrm{Re}(\mathrm{Li}_1(e_5^2)); \ \mathrm{Re}(\mathrm{Li}_1(e_7^k))\Big|_{k=1}^{3}; \mathrm{Re}(\mathrm{Li}_1(e_8));$$

$$\mathrm{Re}(\mathrm{Li}_1(e_9)), \mathrm{Re}(\mathrm{Li}_1(e_9^2)); \ \mathrm{Re}(\mathrm{Li}_1(e_{11}^k))\Big|_{k=1}^{5}; \mathrm{Re}(\mathrm{Li}_1(e_{12})); \ \mathrm{Re}(\mathrm{Li}_1(e_{13}))|_{k=1}^{6};$$

$$\mathrm{Re}(\mathrm{Li}_1(e_{15})), \mathrm{Re}(\mathrm{Li}_1(e_{15}^2)); \mathrm{Re}(\mathrm{Li}_1(e_{16})), \mathrm{Re}(\mathrm{Li}_1(e_{16}^3)); \ \mathrm{Re}(\mathrm{Li}_1(e_{17}))|_{k=1}^{8};$$

$$\mathrm{Re}(\mathrm{Li}_1(e_{19}))|_{k=1}^{9}; \mathrm{Re}(\mathrm{Li}_1(e_{20})), \mathrm{Re}(\mathrm{Li}_1(e_{20}^3)); \ldots$$

$w = 2$

l	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Li_2 basis	1	1	2	2	3	2	4	3	4	3	6	3	7	4	5	5	9	4	10	5
Li_2 new	1	0	1	1	2	0	3	1	2	0	5	0	6	0	1	2	8	0	9	1
basis	1	2	3	3	5	5	8	7	10	10	16	12	21	17	21	21	33	23	40	29
Racinet '01	1	1	1	1	2	2	4	4	7	6	10									
new elements	1	0	1	1	2	1	4	3	5	4	10	5	14	8	12	12	24	11	30	16

The number of the basis elements spanning the dilogarithms resp. w = 2 cyclotomic harmonic sums at lth roots of unity up to 20.

Basis Elements:

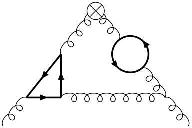
$$\pi;\ \mathrm{Im}(\mathrm{Li}_2(e_3));\ \mathbf{C};\ \mathrm{Im}(\mathrm{Li}_2(e_5)), \mathrm{Im}(\mathrm{Li}_2(e_5^2));\ \mathrm{Li}_4(1/2);\ \mathrm{Im}(\mathrm{Li}_2(e_7^k))\Big|_{k=1}^{3}, \sigma_{1,1}(e_7, e_7^2);$$

$$\mathrm{Im}(\mathrm{Li}_2(e_8)), \sigma_{1,1}(e_8, e_4), \sigma_{1,1}(e_8, e_8^3);\ \mathrm{Im}(\mathrm{Li}_2(e_9)), \mathrm{Im}(\mathrm{Li}_2(e_9^2)), \sigma_{1,1}(e_9, e_9^2),$$

$$\sigma_{1,1}(e_9, e_3), \sigma_{1,1}(e_9^2, e_3);\ \sigma_{1,1}(e_5, e_2), \sigma_{1,1}(e_5^2, e_2), \sigma_{1,1}(e_{10}, e_5), \sigma_{1,1}(e_{10}, e_{10}^3);$$

$$\mathrm{Im}(\mathrm{Li}_2(e_{11}^k))\Big|_{k=1}^{5}, \sigma_{1,1}(e_{11}, e_{11}^k)\Big|_{k=2}^{4}, \sigma_{1,1}(e_{11}^2, e_{11}^k)\Big|_{k=3}^{4}, \ldots$$

Applying cyclotomic HPLs



Using Mellin-Barnes $\rightarrow$ integral over a power series in $x \in [0,1] \xrightarrow{\text{SIGMA}}$ cyclot. S-Sums

$$S_{(2,1,1)}(-x^2;\infty) = \frac{H_{(4,0)}(x)}{x} - 1, \qquad S_{(2,1,1),(1,0,1)}(-x^2,1;\infty) = \frac{H_{(0,0),(4,0)}(x)}{x} - 1$$

$$S_{(2,1,1),(2,-1,1)}(-x^2,1;\infty) = -\frac{1}{x} H_{(4,1),(4,0)}(x)$$

A remaining integral is performed iterating cyclot. HPLs:

$$\int_0^1 dx\, f_{(4,1)}(Kx) H_{\dots}(x), \qquad K \in \left\{\sqrt{1+\kappa}, \frac{1}{\sqrt{1+\kappa}}\right\}, \quad 0 < \kappa \ll 1$$

which yields a generating function in κ for the Mellin moments. Using relations $H_{\dots}(x) \rightarrow H_{\dots}(1/x)$, $H_{\dots}(x) \rightarrow H_{\dots}(x^2)$ the alphabet is reduced to $\{0,-1,1\}$:

$$\{H_{\dots}(1+\kappa), \quad H_{-1,0,0}\left(\sqrt{\kappa+1}\right)\}$$

Nested binomial and inverse binomial sums + generalized cyclotomic sums

Analytic determination of the Nth Taylor coefficient (`Sigma`, `HarmonicSums` [C. Schneider; J. Ablinger])

$$
\begin{aligned}
I \quad = \quad & \frac{1}{45(N+1)\varepsilon^2} - \left[\frac{S_1}{96(N+1)} + \frac{47N^3+20N^2-67N+40}{1800(N-1)N(N+1)}\right]\frac{1}{\varepsilon} \\
& + \frac{105N^3-175N^2+56N+96}{13440(N+1)^2(2N-3)(2N-1)4^N}\binom{2N}{N}\left[\sum_{j=1}^{N}\frac{4^j S_1(j)}{\binom{2j}{j}j^2} - \sum_{j=1}^{N}\frac{4^j}{\binom{2j}{j}j^3} - 7\zeta_3\right] \\
& \frac{(5264N^3-2409N^2-12770N+3528)S_1}{100800(N+1)^2)(2N-3)(2N-1)} + \frac{S_1^2+S_2+3\zeta_2}{360(N+1)} \\
& + \frac{S_3-S_{2,1}+7\zeta_3}{420(N+1)} + \frac{P_1(N)}{2268000(N-1)^2N^2(N+1)^3(2N-3)(2N-1)}
\end{aligned}
$$

$$\sum_{i=1}^{N} \frac{4^i}{i} \frac{1}{\binom{2i}{i}} = \int_0^1 \frac{dz}{\sqrt{1-z}} \frac{z^N - 1}{z - 1}$$

$$\sum_{i=1}^{N} \frac{4^i}{i+1} \frac{1}{\binom{2i}{i}} = \int_0^1 dz \left\{ \frac{z}{\sqrt{1-z}} + \frac{z}{2} \ln\left[\frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right] \right\} \frac{z^N - 1}{z - 1}$$

$$\sum_{i=1}^{N} \frac{4^i}{i} \frac{1}{\binom{2i}{i}} \left[S_1(i-1) - S_1(2i-1)\right] = \frac{1}{2} \int_0^1 \frac{dz}{\sqrt{1-z}} \left[\ln(z) - 2\ln(2)\right] \frac{z^N - 1}{z - 1}$$

$$\sum_{i=1}^{N} \frac{4^i}{i} \frac{1}{\binom{2i}{i}} S_1(i-1) = \int_0^1 \frac{dz}{\sqrt{1-z}} \left[\ln(1-z) - \ln(z) + 2\ln(2)\right] \frac{z^N - 1}{z - 1}$$

$$\sum_{i=1}^{N} \frac{4^i}{i} \frac{1}{\binom{2i}{i}} S_2(i-1) = \int_0^1 \frac{dz}{\sqrt{1-z}} \left\{ \ln^2\left(\frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right) - \zeta_2 \right\} \frac{z^N - 1}{z - 1}$$

- ▷ weighted Mellin transform by $1/\sqrt{1-z}$
- ▷ HPL's over $\{1/x, 1/(1+x), 1/(1-x)\}$ also at argument $1/\sqrt{1-z}$

The New Sums:

Emergence of new nested sums :

$$\sum_{i=1}^{N} \binom{2i}{i}(-2)^i \sum_{j=1}^{i} \frac{1}{j\binom{2j}{j}} S_{1,2}\left(\tfrac{1}{2},-1;j\right)$$

$$= \int_0^1 dx \frac{x^N-1}{x-1}\sqrt{\frac{x}{8+x}}\left[\mathrm{H}^*_{w_{17},-1,0}(x) - 2\mathrm{H}^*_{w_{18},-1,0}(x)\right]$$

$$+\frac{\zeta_2}{2}\int_0^1 dx \frac{(-x)^N-1}{x+1}\sqrt{\frac{x}{8+x}}\left[\mathrm{H}^*_{12}(x) - 2\mathrm{H}^*_{13}(x)\right] + c_3\int_0^1 dx \frac{(-8x)^N-1}{x+\frac{1}{8}}\sqrt{\frac{x}{1-x}}\,,$$

$$w_{12} = \frac{1}{\sqrt{x(8-x)}}, \qquad w_{13} = \frac{1}{(2-x)\sqrt{x(8-x)}},$$

$$w_{17} = \frac{1}{\sqrt{x(8+x)}}, \qquad w_{18} = \frac{1}{(2+x)\sqrt{x(8+x)}}\,.$$

$\sim$ 100 associated independent nested sums.

The associated iterated integrals request root-valued alphabets with about 30 new letters.

J. Ablinger, J. Bümlein, J. Raab, C. Schneider 2013.

Elliptic Integrals

N-space always provides simpler results :

$$T(y) = \int_x^1 \frac{dy}{y} \frac{1}{\sqrt{1-y}} \frac{1}{1-\frac{x}{y}} = \frac{1}{\sqrt{1-x}} \otimes \frac{1}{\sqrt{1-x}}$$

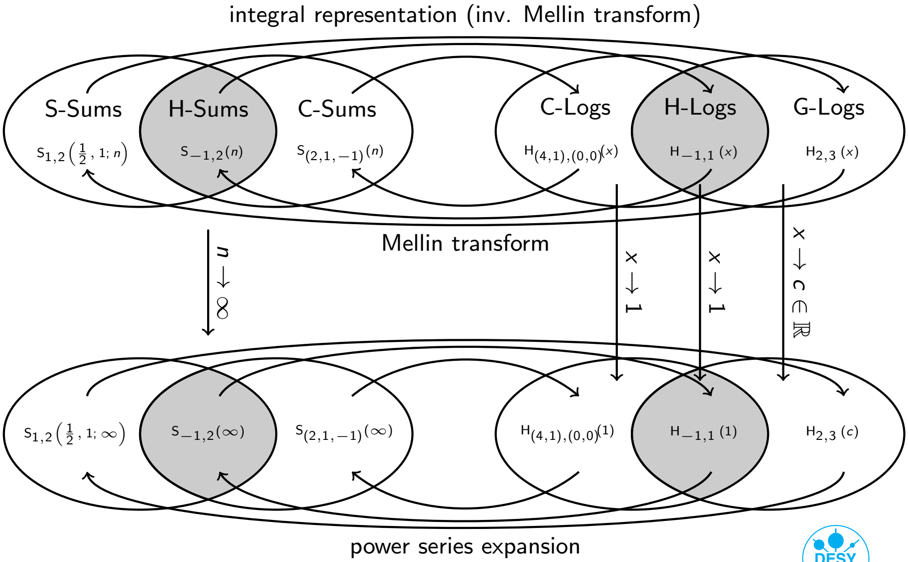
$$= 2i \left[\mathbf{F}\left(arcsin\left(\frac{1}{\sqrt{x}}\right), x\right) - \mathbf{K}(x) \right]$$

... a complicated result in x space.

$$\mathbf{M}[T(y)](N) = \int_0^1 dy y^N T(y) = \frac{4^{2N}}{\binom{2N}{N}^2 (N+\frac{1}{2})^2}$$

... a rather compact result in N-space.

Recommendation: Work in N-Space.


integral representation (inv. Mellin transform)
S-Sums
H-Sums
C-Sums
C-Logs
H-Logs
G-Logs
$S_{1,2}\left(\frac{1}{2},1;n\right)$
$S_{-1,2}(n)$
$S_{(2,1,-1)}(n)$
$H_{(4,1),(0,0)}(x)$
$H_{-1,1}(x)$
$H_{2,3}(x)$
Mellin transform
$n \to \infty$
$x \to 1$
$x \to 1$
$x \to c \in \mathbb{R}$
$S_{1,2}\left(\frac{1}{2},1;\infty\right)$
$S_{-1,2}(\infty)$
$S_{(2,1,-1)}(\infty)$
$H_{(4,1),(0,0)}(1)$
$H_{-1,1}(1)$
$H_{2,3}(c)$
power series expansion

Conclusions

- The mathematical functions expressing Feynman diagrams in N-space form a Hierarchy starting with rational functions, harmonic sums, generalized harmonic sums, cyclotomic sums, their generalization, binomially weighted generalized cyclotomic sums, etc.
- Accordingly, the coresponding iterated integrals and special numbers are organized.
- The cyclotomic polynomials provide a natural extensions of the letters used with iterated integrals leading to harmonic polylogarithms. Corresponding terms occur in massive higher order calculations.
- Via a Mellin transform the cyclotomic harmonic sums are associated, together with the corresponding values for $N \to \infty$. All these systems obey (quasi)shuffle algebras.
- Different special relations of cyclotomic harmonic sums were derived, including differentiation to N. In all cases the asymptotic representations can be generated analytically.
- A large number of new constants emerges for the infinite sums, beyond the MZVs. They were derived and reduced to bases for the real representations.

Conclusions

- Infinite harmonic sums with weights at roots of unity were investigated. They obey more special relations than the case for real representations.
- Nested sums containing binomial weights lead to vst extensions of the alphabets in case of iterated integrals introducing root-valued letters $1/\sqrt{(x-a)(x-b)}; 1/(x-a)/\sqrt{x-b}; a, b \in \mathbb{Q}$.