

ANALYTIC ε -EXPANSION OF THE SCALAR
ONE-LOOP BHABHA BOX FUNCTION * **

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We derive the first three terms of the ε -expansion of the scalar one-loop Bhabha box function from a representation in terms of three generalized hypergeometric functions, which is valid in arbitrary dimensions.

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1. Introduction

One of the important problems in perturbative calculations is a precise determination of the cross section for Bhabha scattering. For this one has to determine the electroweak one-loop corrections in the Standard Model and some parametric enhanced contributions plus the complete photonic corrections to even higher orders. Here we are interested in a determination of photonic $\mathcal{O}(\alpha^2)$ corrections for this process in $d = 4 - 2\varepsilon$, $\varepsilon \rightarrow 0$, dimensions with account of the electron mass m as a regulator of infrared singularities. These corrections naturally concern the virtual two-loop matrix element, which contributes to the cross section due to its interference with the Born matrix element. Of the same order is the absolute square of the one-loop amplitude M_1 . The corresponding cross section contributions have been analytically determined recently [1].

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A peculiarity of the contribution from M_1 is the necessity to determine this amplitude as a function of the parameter ε up to terms of order ε : $M_1(\varepsilon) = m_1/\varepsilon + m_0 + m_1\varepsilon$. In a series of papers, the possibility was studied to find some closed analytical expressions for one-loop 2-, 3- and 4-point functions for arbitrary dimension, external momenta and masses (in principle also complex ones) in terms of generalized hypergeometric functions with relatively simple integral representations [2–4]. The results immediately apply to a deduction of the coefficient m_1 . Beyond that this representation is in particular of great importance for the development of efficient algorithms for the calculation of 5-, 6- and higher point functions since these functions may be reduced to 4- and lower point functions with “unphysical” external kinematics.

In this contribution, we explicitly perform the ε -expansion of the most complicated part: The scalar one-loop box function I_{1111} with two photons (taken here in the s -channel), as it is needed for the calculation of Bhabha scattering up to order $\mathcal{O}(\varepsilon)$. Our starting point is the analytical expression as known from [3, 4]:

$$\begin{aligned}
& \frac{(t - 4m^2)}{\Gamma(2 - \frac{d}{2})} I_{1111}^{(d)} \\
&= \frac{t - 4m^2}{i\pi^{d/2} \Gamma(2 - \frac{d}{2})} \int \frac{d^d k_1}{k_1^2(k_1^2 + 2q_4 k_1)(k_1 + q_1 + q_4)^2(k_1^2 - 2q_3 k_1)} \\
&= -\frac{4m^{d-4}}{s} F_2 \left(\frac{d-3}{2}, 1, 1, \frac{3}{2}, \frac{d-2}{2}; \frac{t}{t-4m^2}, -m^2 Z \right) \\
&\quad + \frac{4m^{d-4}}{(d-3)s} F_{1;1;0}^{1;2;1} \left[\begin{matrix} \frac{d-3}{2}, \frac{d-3}{2}, 1, 1 \\ \frac{d-1}{2}, \frac{d-2}{2}, - \end{matrix}; -m^2 Z, 1 - \frac{4m^2}{s} \right] \\
&\quad - \frac{\sqrt{\pi}(-s)^{\frac{d-4}{2}}}{2^{d-4} m \sqrt{s}} \frac{\Gamma(\frac{d-2}{2})}{\Gamma(\frac{d-1}{2})} F_1 \left(\frac{d-3}{2}, 1, \frac{1}{2}; \frac{d-1}{2}; \frac{sZ}{4}, 1 - \frac{s}{4m^2} \right)
\end{aligned} \tag{1.1}$$

with $Z = \frac{4u}{s(4m^2-t)}$, $q_i^2 = m^2$, $(q_1 + q_4)^2 = s$, $(q_1 + q_2)^2 = t$ and s, t, u being the usual Mandelstam variables. Here F_1, F_2 are Appell hypergeometric functions

$$F_1 \left(\frac{d-3}{2}, 1, \frac{1}{2}, \frac{d-1}{2}; x, y \right) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\left(\frac{d-3}{2}\right)_{r+s}}{\left(\frac{d-1}{2}\right)_{r+s}} \frac{\left(\frac{1}{2}\right)_s}{(1)_s} x^r y^s, \tag{1.2}$$

$$F_2 \left(\frac{d-3}{2}, 1, 1, \frac{3}{2}, \frac{d-2}{2}; x, y \right) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\left(\frac{d-3}{2}\right)_{r+s}}{\left(\frac{3}{2}\right)_r \left(\frac{d-2}{2}\right)_s} x^r y^s \tag{1.3}$$

and the Kampé de Fériet function (KdF) [5] is defined as

$$F_{1;1;0}^{1;2;1} \left[\begin{matrix} \frac{d-3}{2}; & \frac{d-3}{2}, 1; & 1; \\ \frac{d-1}{2}; & \frac{d-2}{2}; & -; \end{matrix} ; x, y \right] = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\left(\frac{d-3}{2}\right)_{r+s}}{\left(\frac{d-1}{2}\right)_{r+s}} \frac{\left(\frac{d-3}{2}\right)_r}{\left(\frac{d-2}{2}\right)_r} x^r y^s. \quad (1.4)$$

The ε -expansion of the generalized hypergeometric functions occurring above is not quite straight forward. In particular since there stands the factor $\Gamma\left(2 - \frac{d}{2}\right) \sim \frac{1}{\varepsilon}$ in front of all of them, one needs their expansion up to order ε^2 . We have to develop different techniques for each of them.

2. Expansion of F_1

We need to know the expansion

$$F_1 \left(\frac{d-3}{2}, 1, \frac{1}{2}, \frac{d-1}{2}; x, y \right) = F_1^0 + \varepsilon F_1^1 + \varepsilon^2 F_1^2 + \dots \quad (2.1)$$

with the kinematics: $x = -\frac{u}{t-4m_e^2} < 0, y = 1 - \frac{s}{4m_e^2} < 0, |y| \gg 1$. In two steps we obtain a form in which one of the parameters of $F_1 \sim \varepsilon$. The transformations are the following:

$$F_1 \left(\frac{d-3}{2}, 1, \frac{1}{2}, \frac{d-1}{2}; x, y \right) = 2 \frac{\Gamma(\frac{d-1}{2}) \Gamma(\frac{6-d}{2})}{\Gamma(\frac{1}{2}) (-y)^{\frac{d-3}{2}}} {}_2F_1 \left[1, \frac{d-3}{2}, \frac{3}{2}; 1-z \right] + \frac{d-3}{(d-6)(-x)\sqrt{-y}} F_1 \left(\frac{6-d}{2}, 1, \frac{1}{2}, \frac{8-d}{2}; \frac{1}{x}, \frac{1}{y} \right) \quad (2.2)$$

$\left(\frac{x}{y} \equiv z = 1 - \frac{st}{(s-4m_e^2)(t-4m_e^2)}; 0 < z, 1-z < 1\right)$ and

$$F_1 \left(\frac{6-d}{2}, 1, \frac{1}{2}, \frac{8-d}{2}; \frac{1}{x}, \frac{1}{y} \right) = \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{8-d}{2})}{\Gamma(\frac{7-d}{2})} y^{\frac{6-d}{2}} {}_2F_1 \left[1, \frac{6-d}{2}, \frac{7-d}{2}; \frac{1}{z} \right] + \frac{(d-6)\sqrt{X-1}(Y-X)}{\sqrt{X}} F_1 \left(1, \frac{d-4}{2}, 1, \frac{3}{2}; X, Y \right) \quad (2.3)$$

with $X = 1 - y = \frac{s}{4m_e^2} \gg 1, Y = \frac{y-1}{x-1} = 1 - \frac{t}{4m_e^2} \gg 1$ ($X > Y, 1 - \frac{1}{Y} = \omega$). Here we observe that the argument of the ${}_2F_1$ as well as those of the F_1 function are larger than 1, i.e. both functions are complex and the imaginary parts must cancel since the F_1 on the l.h.s. has arguments less than 0 and thus is real. For the imaginary part of the F_1 function we obtain

$$\text{Im } F_1 \left(1, \frac{d-4}{2}, 1, \frac{3}{2}; X, Y \right) = \frac{\pi^{\frac{3}{2}} \sqrt{X} (X-1)^{\frac{3-d}{2}}}{2Y \Gamma(\frac{5-d}{2}) \Gamma(\frac{d-2}{2})} {}_2F_1 \left[1, \frac{d-3}{2}, \frac{d-2}{2}; z \right]. \quad (2.4)$$

Transforming ${}_2F_1\left[1, \frac{6-d}{2}, \frac{7-d}{2}; \frac{1}{z}\right]$ with argument $\frac{1}{z} > 1$ to a ${}_2F_1$ function with argument $z < 1$, one shows that the imaginary parts cancel. Transforming (the real part) further to the argument $1 - z$ one finally obtains

$$\begin{aligned} F_1\left(\frac{d-3}{2}, 1, \frac{1}{2}, \frac{d-1}{2}; x, y\right) &= -(d-3) \frac{Y}{\sqrt{X}} \operatorname{Re} F_1\left(1, \frac{d-4}{2}, 1, \frac{3}{2}; X, Y\right) \\ &+ (d-3) \frac{\Gamma(\frac{d-3}{2})\Gamma(\frac{6-d}{2})}{\Gamma(\frac{1}{2})(-y)^{\frac{d-3}{2}}} \sin^2\left(\frac{\pi d}{2}\right) {}_2F_1\left[1, \frac{d-3}{2}, \frac{3}{2}; 1-z\right] \\ &- (d-3) \frac{\pi}{2} \sin\left(\frac{\pi d}{2}\right) (-x)^{-\frac{d-4}{2}} \frac{1}{\sqrt{-y(1-z)}}. \end{aligned} \tag{2.5}$$

To expand up to the required order ($\sim \varepsilon^2$), it is sufficient to set $\sin\left(\frac{\pi d}{2}\right) = -\pi\varepsilon$ and

$${}_2F_1\left[1, \frac{d-3}{2}, \frac{3}{2}; 1-z\right] = \frac{1}{2\sqrt{1-z}} \ln\left(\frac{1+\sqrt{1-z}}{1-\sqrt{1-z}}\right) + O(\varepsilon). \tag{2.6}$$

This simplifies the expansion considerably and the $\operatorname{Re} F_1\left(1, \frac{d-4}{2}, 1, \frac{3}{2}; X, Y\right)$ we take from [4]. Characteristic variables appearing in the result are

$$A = \frac{\sqrt{1-\frac{1}{X}}-1}{\sqrt{1-\frac{1}{X}}+1} < 0, \quad B = \frac{\sqrt{1-\frac{1}{Y}}-1}{\sqrt{1-\frac{1}{Y}}+1} < 0 \tag{2.7}$$

and introducing $a = \sqrt{1-\frac{1}{X}}$, $b = \sqrt{1-\frac{1}{Y}}$ we can write ($1 > a > b > 0$) $A = \frac{a-1}{a+1}$, $B = \frac{b-1}{b+1}$ and

$$F_1^0 = -\frac{m_\varepsilon}{\sqrt{s}} \frac{1}{b} \ln(-B), \tag{2.8}$$

yielding the correct $\frac{1}{\varepsilon}$ -term of D_0 [6]. Keeping only the leading terms, collecting the contributions, yields correspondingly

$$\begin{aligned} &\frac{2}{s(t-4m_\varepsilon^2)} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-\varepsilon)} \Gamma(1-\varepsilon) \left(-\frac{s}{4}\right)^{-\varepsilon} \Gamma(\varepsilon) \frac{1}{b} \left[\operatorname{Re}\{\ln(B) + \dots\}\right. \\ &\left. - \pi^2 \varepsilon^2 \ln\left(\frac{1-AB}{A-B}\right) - \pi^2 \varepsilon \left(1 + \varepsilon \ln\left(\frac{X}{Y} - 1\right)\right) + O(\varepsilon^3)\right], \end{aligned} \tag{2.9}$$

where the higher order terms in ε of $\operatorname{Re}\{\ln(B) + \dots\}$ have to be taken from [4].

3. Expansion of F_2

We need to know the expansion

$$F_2\left(\frac{d-3}{2}, 1, 1, \frac{3}{2}, \frac{d-2}{2}; \omega, z\right) = F_2^0 + \varepsilon F_2^1 + \varepsilon^2 F_2^2 + \dots \tag{3.1}$$

with the kinematics: $\omega = \frac{t}{t-4m_e^2}, z = -4m_e^2\left(\frac{1}{s} + \frac{1}{t-4m_e^2}\right)$. At first we perform the following Euler transformation:

$$\begin{aligned} &F_2\left(\frac{d-3}{2}, 1, 1, \frac{3}{2}, \frac{d-2}{2}; \omega, z\right) \\ &= (1-z)^{\frac{3-d}{2}} F_2\left(\frac{d-3}{2}, 1, \frac{d-4}{2}, \frac{3}{2}, \frac{d-2}{2}; \frac{\omega}{1-z}, -\frac{z}{1-z}\right). \end{aligned} \tag{3.2}$$

The factor $(1-z)^{-\frac{d-3}{2}}$ will be dropped in what follows and is taken into account again when collecting the results. Introducing $\frac{\omega}{1-z} = \omega'$ and $-\frac{z}{1-z} = z'$, we have

$$0 \leq \omega' + z' < 1. \tag{3.3}$$

With $\alpha = \frac{1}{2} - \varepsilon, \beta = 1, \beta' = -\varepsilon, \gamma = \frac{3}{2}$ and $\gamma' = 1 - \varepsilon$

$$F_2(\alpha, \beta, \beta', \gamma, \gamma', \omega', z') = {}_2F_1(\alpha, \beta, \gamma; \omega') + \beta' S(\alpha, \beta, \beta', \gamma, \omega', z'), \tag{3.4}$$

where $\gamma' = 1 + \beta'$ has been used and

$$S(\alpha, \beta, \beta', \gamma, \omega', y) = \sum_{n=1}^{\infty} \frac{(\alpha)_n}{\beta' + n} \frac{y^n}{n!} {}_2F_1(\alpha + n, \beta, \gamma; \omega'). \tag{3.5}$$

In order to get rid of the denominator $\beta' + n$ we differentiate S w.r.t. y and use [7]

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n (\delta)_n}{n! (\delta')_n} y^n {}_2F_1(\alpha + n, \beta, \gamma; \omega') = F_2(\alpha, \beta, \delta, \gamma, \delta'; \omega', y) \tag{3.6}$$

with $\delta = \delta'$. Applying again the Euler relation,

$$\begin{aligned} F_2(\alpha, \beta, \delta, \gamma, \delta; \omega', y) &= (1-y)^{-\alpha} F_2\left(\alpha, \beta, 0, \gamma, \delta; \frac{\omega'}{1-y}, -\frac{y}{1-y}\right) \\ &= (1-y)^{-\alpha} {}_2F_1\left(\alpha, \beta, \gamma; \frac{\omega'}{1-y}\right), \end{aligned} \tag{3.7}$$

we finally have ($\beta = 1, \gamma = \frac{3}{2}$ inserted)

$$\begin{aligned} S(\alpha, \beta', \omega', z') &= \int_{y=0}^{z'} \frac{\partial S(\alpha, \beta', \omega', y)}{\partial y} dy \\ &= S_0\left(\frac{1}{2}, 0, \omega', z'\right) + \varepsilon S_1\left(\frac{1}{2}, 0, \omega', z'\right) + O(\varepsilon^2), \end{aligned} \quad (3.8)$$

with

$$\begin{aligned} \frac{\partial S}{\partial y} &= \frac{1}{y} \left[\frac{1}{(1-y)^\alpha} {}_2F_1\left(\alpha, 1, \frac{3}{2}; \frac{\omega'}{1-y}\right) - {}_2F_1\left(\alpha, 1, \frac{3}{2}; \omega'\right) \right] \\ &\quad + \frac{\varepsilon}{y} S\left(\frac{1}{2}, 0, \omega', y\right) + O(\varepsilon^2). \end{aligned} \quad (3.9)$$

To complete the ε -expansion, we need (with $z = \frac{\omega'}{1-y}$)

$$\begin{aligned} {}_2F_1\left(\alpha, 1, \frac{3}{2}; z\right) &= {}_2F_1\left(\frac{1}{2}, 1, \frac{3}{2}; z\right) \\ &\quad + \varepsilon \delta^{(1)} F(z, u) + \varepsilon^2 \delta^{(2)} F(z, u) + \dots \end{aligned} \quad (3.10)$$

with the abbreviation $u = \frac{1+\sqrt{z}}{1-\sqrt{z}}$. Explicitly $u = \frac{\sqrt{1-y+w}}{\sqrt{1-y-w}}$ with $w = \sqrt{\omega'}$,

$$\delta^{(1)} F(z, u) = \frac{1}{2\sqrt{z}} \left[2\text{Li}_2\left(-\frac{1}{u}\right) - 2\ln(u)\ln(1+u) + \frac{3}{2}\ln^2(u) + \zeta(2) \right] \quad (3.11)$$

and

$$\begin{aligned} \delta^{(2)} F(z, u) &= \frac{1}{2\sqrt{z}} \left[-4S_{1,2}\left(-\frac{1}{u}\right) - \left(\ln(u) + 2\ln\left(1 + \frac{1}{u}\right)\right) \zeta(2) \right. \\ &\quad - 4\ln\left(1 + \frac{1}{u}\right) \text{Li}_2\left(-\frac{1}{u}\right) + 2\ln(u)\ln^2\left(1 + \frac{1}{u}\right) + \ln^2(u)\ln\left(1 + \frac{1}{u}\right) \\ &\quad \left. + \frac{1}{6}\ln^3(u) + 2\zeta(3) + 2\text{Li}_3\left(-\frac{1}{u}\right) \right]. \end{aligned} \quad (3.12)$$

3.1. Order ε of F_2

In this order we have

$$\begin{aligned} F_2(\alpha, \beta, \beta', \gamma, \gamma', \omega', z') &= {}_2F_1\left(\frac{1}{2}, 1, \frac{3}{2}; \omega'\right) \\ &\quad + \varepsilon \delta^{(1)} F(\omega', u_0) - \varepsilon S\left(\frac{1}{2}, 0, \omega', z'\right) + O(\varepsilon^2), \\ &\equiv F_2^0 + \varepsilon F_2^1 + O(\varepsilon^2), \end{aligned} \quad (3.13)$$

where the “scale” $u_0 = u(y = 0) = \frac{1+w}{1-w} \sim \frac{s}{m_e^2} \frac{1}{1-\frac{4m_e^2}{t}}$. u_0 is large for $s \gg m_e^2$ and $-t \gg 4m_e^2$ and sets the scale for the variable u in general. Further we introduce $u_1 = u(y = z') < u_0$. Thus we can write

$$\begin{aligned}
 F_2^0 &= \frac{1}{2w} \ln(u_0), \\
 S\left(\frac{1}{2}, 0, \omega', y\right) &= \frac{1}{2w} \int_{y=0}^y \frac{1}{y} \ln\left(\frac{u}{u_0}\right) dy \\
 &= \frac{1}{2w} \int_u^{u_0} \left[\frac{2}{u-1} - \frac{1}{u-u_0} - \frac{1}{u-\frac{1}{u_0}} \right] \ln\left(\frac{u}{u_0}\right) du \\
 &\equiv \frac{1}{2w} S_0(u_0, u)
 \end{aligned} \tag{3.14}$$

and $S(\frac{1}{2}, 0, \omega', z') = \frac{1}{2w} S_0(u_0, u_1)$. The integration yields:

$$\begin{aligned}
 S_0(u_0, u) &= 2\text{Li}_2\left(\frac{1}{u}\right) + \text{Li}_2\left(\frac{u}{u_0}\right) - \text{Li}_2\left(\frac{1}{u_0 u}\right) + 2\text{Li}_2\left(-\frac{1}{u_0}\right) - \zeta(2) \\
 &- \ln\left(\frac{u}{u_0}\right) \left[2\ln(u-1) - \ln(u_0 u - 1) - \ln\left(1 - \frac{u}{u_0}\right) - \frac{1}{2} \ln\left(\frac{u}{u_0}\right) \right].
 \end{aligned} \tag{3.15}$$

3.2. Order ε^2 of F_2

The next order can be written in the form

$$F_2^2 = \delta^{(2)} F(\omega', u_0) - S_1\left(\frac{1}{2}, 0, \omega', z'\right) \tag{3.16}$$

with

$$\begin{aligned}
 S_1\left(\frac{1}{2}, 0, \omega', z'\right) &\equiv S_1(u_0, u_1) = \int_{y=0}^{z'} \frac{dy}{y} \left\{ \frac{1}{2w} \ln(1-y) \ln(u) \right. \\
 &\left. + \delta^{(1)} F(\omega', u) - \delta^{(1)} F(\omega', u_0) + S\left(\frac{1}{2}, 0, \omega', y\right) \right\}.
 \end{aligned} \tag{3.17}$$

The curly bracket in the above integrand finally reads

$$\{\dots\} = \text{Li}_2\left(\frac{u}{u_0}\right) - \text{Li}_2(1) + \text{Li}_2\left(\frac{1}{u^2}\right) - \text{Li}_2\left(\frac{1}{u_0 u}\right) + 2\ln(u) \ln\left(\frac{u_0 - 1}{u - 1}\right)$$

$$\begin{aligned}
& -2\ln(u_0 + 1)\ln\left(\frac{u}{u_0}\right) + \frac{3}{2}\ln(u_0u)\ln\left(\frac{u}{u_0}\right) - \ln\left(\frac{u}{u_0}\right) \\
& \times \left[2\ln(u - 1) - \ln(u_0u - 1) - \ln\left(1 - \frac{u}{u_0}\right) - \frac{1}{2}\ln\left(\frac{u}{u_0}\right) \right]. \quad (3.18)
\end{aligned}$$

There is no problem to perform the final integration, but the expressions blow up considerably. Therefore we confine ourselves here to the leading terms only by considering u (u_0) as large and drop the small quantities. Then the integral can be written in the simplified form ($\frac{u}{u_0} = v$ the new integration variable)

$$\begin{aligned}
S_1(u_0, u_1) &= \frac{1}{2w} \int_{v=r}^1 \left[\frac{1}{v} + \frac{1}{1-v} \right] \left\{ \text{Li}_2(v) - \text{Li}_2(1) - \ln^2(v) - \ln(u_0)\ln(v) \right. \\
& \left. + \ln(v)\ln(1-v) \right\} = \frac{1}{2w} \left[-\text{Li}_3(1-r) + \ln(1-r)\text{Li}_2(r) \right. \\
& \left. + (\ln(u_0) + \ln(1-r))\text{Li}_2(1-r) - \ln(r)\text{Li}_2(r) + \frac{1}{3}\ln^3(r) \right. \\
& \left. + \left(\frac{1}{2}\ln(u_0) - \ln(1-r) \right) \ln^2(r) + \ln^2(1-r)\ln(r) - \zeta(2)\ln\left(\frac{1}{r} - 1\right) \right] \quad (3.19)
\end{aligned}$$

with $0 < r = \frac{u_1}{u_0} < 1$. If one wants higher precision, it is easier to expand in $\frac{1}{u}$ ($\frac{1}{u_0}$) instead of performing all integrals analytically, which is possible nevertheless. Collecting the results, we have

$$-\frac{2(m_e^2)^{-\varepsilon}}{s(t-4m_e^2)} (1-z)^\varepsilon \Gamma(\varepsilon) \frac{1}{\sqrt{\omega}} (\ln(u_0) + \dots). \quad (3.20)$$

4. Expansion of the Kampé de Fériet function

We need to know the expansion

$$F_{1;1;0}^{1;2;1} \left[\begin{matrix} \frac{d-3}{2}; & \frac{d-3}{2}, & 1; & 1; \\ \frac{d-1}{2}; & & \frac{d-2}{2}; & -; \end{matrix} \middle| x, y \right] = K^0 + \varepsilon K^1 + \varepsilon^2 K^2 + \dots \quad (4.1)$$

with the kinematics: $x = -4m_e^2 \left(\frac{1}{s} + \frac{1}{t-4m_e^2} \right)$, $y = 1 - \frac{4m_e^2}{s}$. In this case we begin with the integral representation of the KdF function:

$$\begin{aligned}
& F_{1;1;0}^{1;2;1} \left[\begin{matrix} \frac{d-3}{2}; & \frac{d-3}{2}, & 1; & 1; \\ \frac{d-1}{2}; & & \frac{d-2}{2}; & -; \end{matrix} \middle| x, y \right] \\
& = \frac{d-3}{2} \int_0^1 \frac{dt t^{\frac{d-5}{2}}}{1-t y} {}_2F_1 \left(1, \frac{d-3}{2}, \frac{d-2}{2}, x t \right). \quad (4.2)
\end{aligned}$$

Again we perform a shift such that one of the parameters of the ${}_2F_1 \sim \varepsilon$:

$$\begin{aligned} {}_2F_1\left(1, \frac{d-3}{2}, \frac{d-2}{2}, x t\right) &= (1-x t)^{-\frac{1}{2}} {}_2F_1\left(-\varepsilon, \frac{1}{2}, 1-\varepsilon, x t\right) \\ &= (1-x t)^{-\frac{1}{2}} [1-\varepsilon S(x t)] \end{aligned} \tag{4.3}$$

with

$$S(x t) = \sum_{n=1}^{\infty} \frac{1}{n-\varepsilon} \frac{\left(\frac{1}{2}\right)_n}{n!} (x t)^n. \tag{4.4}$$

The ε -expansion of $S(x t)$ can again be obtained by first differentiating S :

$$\begin{aligned} \frac{\partial S}{\partial x} &= \frac{1}{x} \left(\frac{1}{\sqrt{1-x t}} - 1 \right) + \frac{\varepsilon}{x} \sum_{n=1}^{\infty} \frac{1}{n} \frac{\left(\frac{1}{2}\right)_n}{n!} (x t)^n + O(\varepsilon^2) \\ &= \frac{1}{x} \left(\frac{1}{\sqrt{1-x t}} - 1 \right) + \frac{\varepsilon}{x} S(x t)|_{\varepsilon=0} + O(\varepsilon^2). \end{aligned} \tag{4.5}$$

In order to obtain S to $O(\varepsilon)$, we need the following integrals:

$$S(x t)|_{\varepsilon=0} = \int_0^x \frac{dx}{x} \left(\frac{1}{\sqrt{1-x t}} - 1 \right) = 2 \ln \left(1 + \frac{1}{v} \right) \tag{4.6}$$

and

$$\int_0^x \frac{dx}{x} S(x t)|_{\varepsilon=0} = -2 \text{Li}_2 \left(-\frac{1}{v} \right) - 2 \ln^2 \left(1 + \frac{1}{v} \right), \tag{4.7}$$

where we introduced $v = \frac{1+\sqrt{1-x t}}{1-\sqrt{1-x t}}$. Thus the above integral reads

$$\int_0^1 \frac{dt t^{-\frac{1}{2}}}{1-t y} \frac{1}{\sqrt{1-x t}} \left\{ 1 - \varepsilon \ln \left(\frac{4}{v x} \right) - \varepsilon^2 \left[-2 \text{Li}_2 \left(-\frac{1}{v} \right) - \frac{1}{2} \ln^2 \left(\frac{4}{v x} \right) \right] \right\}. \tag{4.8}$$

After a variable transformation we can write

$$\begin{aligned} \int_0^1 \frac{dt t^{-\frac{1}{2}}}{1-t y} \frac{1}{\sqrt{1-x t}} f(v) &= -\frac{1}{\sqrt{y-x}} \int_0^1 dt \left\{ b_1 \left[\frac{1}{1+b_1 t} + \frac{1}{1-b_1 t} \right] \right. \\ &\quad \left. - b_2 \left[\frac{1}{1+b_2 t} + \frac{1}{1-b_2 t} \right] \right\} f \left(\frac{v_1}{t^2} \right) \end{aligned} \tag{4.9}$$

with $v_1 = v(t = 1) = \frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}$, $b_1 = \frac{1}{\sqrt{v_0 v_1}}$ and $b_2 = \sqrt{\frac{v_0}{v_1}}$ with $v_0 = \frac{1+\sqrt{1-\frac{x}{y}}}{1-\sqrt{1-\frac{x}{y}}}$. v_0 and v_1 both being large, results in $b_1 \ll 1$ and $b_2 < 1$ but very close to 1. Taking again the attitude to keep only leading contributions, the b_1 -contribution can be dropped. The Li_2 -function in the second order term can be written as

$$\text{Li}_2\left(-\frac{t^2}{v_1}\right) = 2 \left[\text{Li}_2\left(i\frac{t}{\sqrt{v_1}}\right) + \text{Li}_2\left(-i\frac{t}{\sqrt{v_1}}\right) \right] \quad (4.10)$$

so that integration is possible. We do get, however, relatively complicated complex conjugate contributions. On the other hand since $v_1 \gg 1$ this contribution is small from the very beginning and can be well approximated by expanding the Li_2 -function. Here it is dropped altogether. Thus we are left with the following contributions:

$$K^0 = \frac{d-3}{2\sqrt{\omega}} \ln(u_0), \quad (4.11)$$

$$K^1 = \frac{d-3}{2\sqrt{\omega}} \left[\ln\left(\frac{1+b_2}{1-b_2}\right) \ln\left(\frac{v_1 x}{4}\right) + 2(\text{Li}_2(b_2) - \text{Li}_2(-b_2)) \right], \quad (4.12)$$

$$K^2 = \frac{d-3}{2\sqrt{\omega}} \left[\frac{1}{2} \ln\left(\frac{1+b_2}{1-b_2}\right) \ln^2\left(\frac{v_1 x}{4}\right) + 2 \ln\left(\frac{v_1 x}{4}\right) (\text{Li}_2(b_2) - \text{Li}_2(-b_2)) + 4(\text{Li}_3(b_2) - \text{Li}_3(-b_2)) \right]. \quad (4.13)$$

Collecting the results we obtain

$$\frac{2(m_e^2)^{-\varepsilon}}{s(t-4m_e^2)} \Gamma(\varepsilon) \frac{1}{\sqrt{\omega}} (\ln(u_0) + \dots). \quad (4.14)$$

We see that the $\frac{1}{\varepsilon}$ -term cancels against the one from the F_2 contribution.

5. Expansion of the scalar box function with Feynman parameters

In order to have an independent check of the above results, we derived a Feynman parameter integral representation for the ε -expansion. We follow closely [8], where the scalar four-point integral was treated with a finite photon mass in $d = 4$ dimensions.

The function to be calculated is, in LoopTools notations [9]:

$$\begin{aligned}
 J &= D_0(m^2, m^2, m^2, m^2 \mid t, s \mid m^2, 0, m^2, 0) \\
 &= \frac{(2\pi\mu)^{2\varepsilon}}{i\pi^2} \int \frac{d^d k}{k^2(k^2 + 2kq_4)(k^2 - 2kq_3)(k + q_1 + q_4)^2}. \quad (5.1)
 \end{aligned}$$

A constant transforms the normalization of D_0 to that of $I_{1111}^{(d)}$:

$$D_0 = (4\pi\mu^2)^\varepsilon I_{1111}^{(d)}. \quad (5.2)$$

The infrared singularity may be isolated in a 3-point function C_0 by redefining

$$J = \frac{2}{s} (F + C_0), \quad (5.3)$$

with

$$C_0 = C_0(t, \mu^2, m^2 \mid m^2, \mu^2, 0) = \frac{(2\pi\mu)^{2\varepsilon}}{i\pi^2} \int \frac{d^d k}{k^2(k^2 + 2kq_4)(k^2 - 2kq_3)}, \quad (5.4)$$

and with

$$F = \frac{(2\pi\mu)^{2\varepsilon}}{i\pi^2} \int \frac{d^d k (s/4 - k^2)}{k^2(k^2 + 2kq_4)(k^2 - 2kq_3)(k + q_1 + q_4)^2} \quad (5.5)$$

being a finite scalar four point function.

The ε -expansions may be easily derived now starting from

$$C_0 = \Gamma(\varepsilon) \int_0^1 \frac{dx}{2p_x^2} \left[\frac{4\pi\mu^2}{p_x^2} \right]^\varepsilon, \quad (5.6)$$

$$\begin{aligned}
 F &= \Gamma(2 + \varepsilon) \int_0^1 dx dy dz \frac{y^2 z}{(M^2)^2} \left(\frac{1}{2} y z s + \frac{1 - 2\varepsilon}{1 + \varepsilon} M^2 \right) \left[\frac{4\pi\mu^2}{p_x^2} \right]^\varepsilon \\
 &= \Gamma(2 + \varepsilon) [I_0 + \varepsilon I_1 + \varepsilon I_L] + \dots, \quad (5.7)
 \end{aligned}$$

with

$$p_x^2 = -x(1 - x)t + m^2 - i\epsilon, \quad (5.8)$$

$$M^2 = y[yz^2 p_x^2 - (1 - y)(1 - z)s]. \quad (5.9)$$

Thus, the four-point function may be represented as follows:

$$I_0 = - \int_0^1 \frac{dx}{2p_x^2} \ln(-A), \quad (5.10)$$

$$I_1 = -3 \int_0^1 \frac{dx}{p_x^2} dz \left[\frac{z}{N(z)} + \frac{Az(1-z)}{N(z)^2} \ln \frac{z^2}{(1-z)(-A)} \right], \quad (5.11)$$

with

$$A = \frac{s}{p_x^2}, \quad (5.12)$$

$$N(z) = z^2 + (1-z)A. \quad (5.13)$$

The last function will be given here in short as a three-fold integral. But it is evident that the y -integration leads to simple integrals in terms of dilogarithms or simpler functions:

$$I_L = \int_0^1 \frac{dx}{p_x^2} dz dy \left[\frac{Ay}{2z^2 K(y)^2} + \frac{y}{zK(y)} \right] \ln \frac{4\pi\mu^2 A}{z^2 y K(y) s} \quad (5.14)$$

with

$$K(y) = y - (1-y) \frac{1-z}{z^2} A. \quad (5.15)$$

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